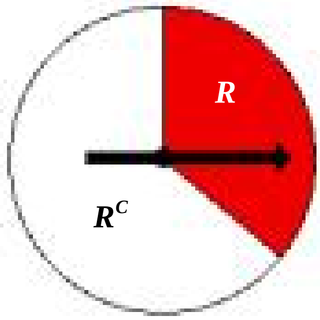
Contemporary Mathematics
Instruction: Binomial Probability

Imagine a game that requires spinning the dial below. Imagine a player that "wins" if the arrow lands on the *R*-sector (occupying one-third of the dial) only once in the two spins.



A natural question asks, "What is the probability that the player wins?" To calculate the probability that the player "wins," we will think of each spin as a trial of an experiment, and we will consider two events, R and R^C. Since *R* occupies one-third of the dial, $P(R) = 1/3$ and $P(R^C) = 2/3$.

Recall that the player "wins" only if the arrow lands on the *R*-sector exactly once in two spins. In other words, the player could win if *R* occurs on the first spin *and R* does not occur on the second spin, *<u>or</u>* if *R* does not occur on the first spin *and R* does occur on the second spin. Let R_1 represent the event that *R* occurs on the first spin, R_1^C represent the event that R^C occurs on the first spin, R_2 represent the event that *R* occurs on the second spin, and R_2^C represent the event that R^C occurs on the second spin. Applying the *Addition Rule of Probability for Mutually Exclusive Events* together with the *Multiplication Rule of Probability of Independent Events* we can find $P(\text{winning})$ as below.

$$P(\text{winning}) = P(R_1) \cdot P(R_2^C) + P(R_2) \cdot P(R_1^C)$$
$$P(\text{winning}) = \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{3}$$
$$P(\text{winning}) = \frac{4}{9}$$

This probability can be written as below.

$$P(\text{winning}) = C(2,1) \cdot \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{2-1}$$

This is a specific example of the *Binomial Probability Formula* stated below in general terms.

> Let $\mathcal{W}$ represent *x* number of occurrences of *R* in *n* trials.
> If $P(R)$ remains constant throughout *n* trials, the probability of $\mathcal{W}$ is given by
> $$P(\mathcal{W}) = C(n,x)\cdot\left[P(R)\right]^x\cdot\left[P\left(R^C\right)\right]^{n-x}.$$

Applying the formula, to the situation above, we note that $n = 2$ because there were two spins, $x = 1$ because the player won only if *R* occurred exactly once, $P(R) = 1/3$, and $P\left(R^C\right) = 2/3$. Thus,

$$P(\mathcal{W}) = C(n,x)\cdot\left[P(R)\right]^x\cdot\left[P\left(R^C\right)\right]^{n-x}$$

$$P(\mathcal{W}) = C(2,1)\cdot\left(\frac{1}{3}\right)^1\cdot\left(\frac{2}{3}\right)^{2-1}$$

$$P(\mathcal{W}) = 2\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{4}{9}$$

Let's consider an example from meteorology. Consider a set of weather conditions such that a meteorologist can determine from past data that there exists a 30% chance for rain (a 0.3 probability for rain). What is the probability, that it will rain at most three times in ten instances of these weather conditions? To answer this question, we first note the phrase "at most" and recall that the *Binomial Probability Formula* applies to an exact number of occurrences. Nevertheless, the *Binomial Probability Formula* will help us arrive at an answer to the question. Let *R* represent the event that it rains under these conditions, and Let $\mathcal{W}_1$ represent exactly one instance of rain in ten instances of the given set of conditions. $P(\mathcal{W}_1)$, then, represents the probability that there is one occurrence of *R* in ten trials:

$$P(\mathcal{W}_1) = C(n,x)\cdot\left[P(R)\right]^x\cdot\left[P\left(R^C\right)\right]^{n-x}$$

$$P(\mathcal{W}_1) = C(10,1)\cdot(0.3)^1\cdot(0.7)^{10-1}$$

$$P(\mathcal{W}_1) = 10\cdot(0.3)\cdot(0.7)^9$$

$$P(\mathcal{W}_1) \approx 0.1210608$$

Now, let $\mathcal{W}_2$ represent exactly two instances of rain and $\mathcal{W}_3$ represent exactly three instances of rain, and find $P(\mathcal{W}_2)$ and $P(\mathcal{W}_3)$ in the same manner.

$$P(\mathcal{W}_2) = C(n,x)\cdot\left[P(R)\right]^x \cdot \left[P\left(R^C\right)\right]^{n-x}$$
$$P(\mathcal{W}_2) = C(10,2)\cdot(0.3)^2 \cdot (0.7)^{10-2}$$
$$P(\mathcal{W}_2) = 45\cdot(0.09)\cdot(0.7)^8$$
$$P(\mathcal{W}_2) \approx 0.2334744$$

$$P(\mathcal{W}_3) = C(n,x)\cdot\left[P(R)\right]^x \cdot \left[P\left(R^C\right)\right]^{n-x}$$
$$P(\mathcal{W}_3) = C(10,3)\cdot(0.3)^3 \cdot (0.7)^{10-3}$$
$$P(\mathcal{W}_3) = 120\cdot(0.027)\cdot(0.7)^7$$
$$P(\mathcal{W}_3) \approx 0.2668279$$

We are now ready to answer the question, "What is the probability, that it will rain at most three times in ten instances of these weather conditions?" In other words, "What is the value of $P(\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3)$?" Applying the *Addition Rule of Probability for Mutually Exclusive Events* we find $P(\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3)$ below.

$$P(\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3) = P(\mathcal{W}_1) + P(\mathcal{W}_2) + P(\mathcal{W}_3)$$
$$P(\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3) \approx 0.1210608 + 0.2334744 + 0.2668279$$
$$P(\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3) \approx 0.6213631$$

Assignment 3.9

Problems

#1 Consider an experiment that requires spinning the dial in Figure 1. What is the probability that the arrow will indicate the number seven exactly five times in nine trials?

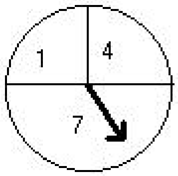


Figure 1

#2 Consider an experiment for which $P(A) = 0.25$. Assume $\mathcal{W}$ represents exactly two occurrences of A in four trials, and evaluate $P(\mathcal{W})$.

#3 Consider an experiment that requires rolling a six-sided die as shown in Figure 2. What is the probability that the five will land face up exactly three times in four rolls?

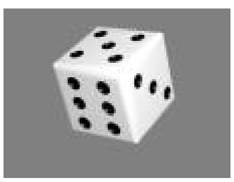


Figure 2

#4 Researchers have determined that patients have only a 0.5% chance to suffer serious side-effects from a drug. If the drug is administered to 100 patients, what is the probability that at most two patients will suffer serious side effects?

#5 Consider an experiment that requires spinning the dial in Figure 3. What is the probability that the arrow will indicate the number 3 *at least* once in five trials?

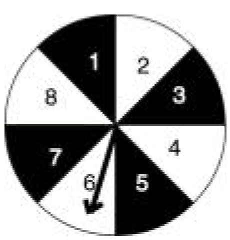


Figure 3

Contemporary Mathematics

Instruction: Expected Value

This lecture discusses *expected value* or mathematical expectation. Expected value is a special sum associated with the probability distribution of a game. A *game*, defined below, is an experiment.

> A *game* is an experiment with a defined set of discrete values called payoffs for a random variable, *X*, that represents a set of mutually exclusive events whose union comprises the sample space of the experiment.

The probability distribution associated with a game lists the values of a random variable, *X*, and the probabilities, *P*(*X*), corresponding to the values of the random variable. The values of the random variable will be "payoffs" measured in dollars, time units, spaces on a board, or other units of measures. The *expected value* associated with a probability distribution equals the sum of the products of the values of the payoffs of the random variable and the probabilities corresponding to those payoffs.

> If X is a discrete random variable with payoffs $X_1, X_2, \ldots, X_n$, occurring with probabilities $P(X_1), P(X_2), \ldots, P(X_n)$, respectively, then the *expected value*, denoted $\mathbb{E}(X)$, is given by the sum of the products:
>
> $$\mathbb{E}(X) = X_1 \cdot P(X_1) + X_2 \cdot P(X_2) + \cdots + X_n \cdot P(X_n)$$

Consider a roulette wheel with thirty-eight equally-likely possible outcomes. Typically, a bet placed on a single number pays thirty-five to one, meaning a winning one-dollar bet earns thirty-five dollars. The table below represents a probability distribution for a one-dollar bet placed on one spin of the roulette wheel.

X	–\$1.00	\$35.00
$P(X)$	$\frac{37}{38}$	$\frac{1}{38}$

The sum of products below gives $\mathbb{E}(X)$, the expected value of this experiment.

$$\mathbb{E}(X) = -1 \cdot \frac{37}{38} + 35 \cdot \frac{1}{38}$$

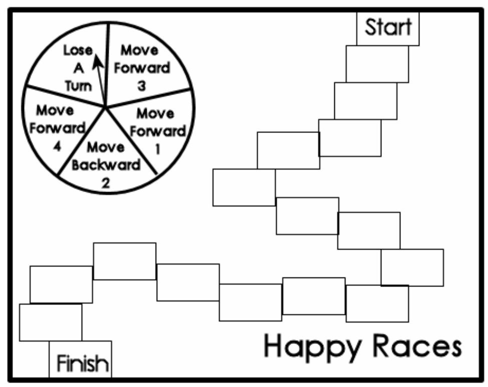
$$\mathbb{E}(X) = -\frac{37}{38} + \frac{35}{38}$$

$$\mathbb{E}(X) = -\frac{2}{38} \approx -0.053$$

Thus, the expected value of the one dollar bet on a single number and single spin of a roullette wheel equals a loss of about five cents.

Significantly, the expected value does not equal the amount the player might win or lose. Accordingly, expected value has nothing to do with the subjective expectations of the player. The player may optimistically expect to win thirty-five dollars, or the player may realistically expect to lose one dollar. No rational player would expect to lose five cents. Indeed, it is impossible to lose five cents on a single spin. The term "expected" comes from the average payoff per game after a large number of trials. The roullette player can expect to lose an average of five cents per game if he/she plays the game a large number of times.

Consider the board game with the unbiased spinner in Figure 1. Assume each player



spins the dial on his or her turn and moves along the board according to the result of the spin. Any adult who plays a game like this may have found him/herself wondering if the game will ever end. In other words, "Can a player reasonably expect to reach the finish space?" The question involves expected value. The expected value associated with a player's turn is calculated below.

$$\mathbb{E}(X) = 3\left(\frac{1}{5}\right) + 1\left(\frac{1}{5}\right) - 2\left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) + 0\left(\frac{1}{5}\right)$$

$$\mathbb{E}(X) = 3(0.2) + 1(0.2) - 2(0.2) + 4(0.2) + 0(0.2)$$

$$\mathbb{E}(X) = 0.6 + 0.2 - 0.4 + 0.8 = 1.2$$

According to the result, a player can expect to average a gain of 1.2 spaces after a large number of turns, so, yes, the player can expect to finish the game.

Another question associated with expected value involves fairness. Previous lectures equated fairness with a lack of bias. If all the outcomes of a die are equally likely the die is said to be fair. With games, however, fairness equates to a zero expected value. Consider a game that pays according to the results of the unbiased spinner below.

Lecture 3.10

The expected value, calculated to be zero below, shows that this game is *fair*.

$$\mathbb{E}(X) = \$8\left(\frac{1}{4}\right) + \$2\left(\frac{1}{4}\right) - 5\left(\frac{1}{2}\right)$$
$$\mathbb{E}(X) = \$8(0.25) + \$2(0.25) - \$5(0.5)$$
$$\mathbb{E}(X) = \$2 + \$0.5 - \$2.5$$
$$\mathbb{E}(X) = \$0$$

Problems

#1 Consider a game where the player spins the number wheel in Figure 1 and loses a dollar amount equal to odd values indicated by the arrow but wins a dollar amount equal to ten times even values indicated by the arrow. What is the expected value of the game?

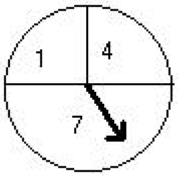


Figure 1

#2 An insurance agency sells an insurance policy that pays $50,000.00 in benefits in a given year if a worker is injured on the job and stays in the hospital one week or more. The policy sells for a monthly premium of $42.00. Qualified workers must work for a company whose safety regulations give the workers only a 0.002 probability of injury. What is the expected value of the policy for the insurance agency?

#3 Consider a game where the player rolls the die shown in Figure 2. Is the game fair if the player wins a dollar amount equal to the number of dots landing face up unless only one dot lands face up, in which case, the player loses twenty dollars?

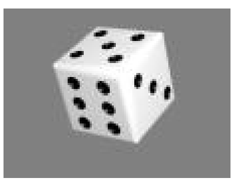


Figure 2

#4 Marketing research developed the probability distribution below where the random variable represents the length of time (rounded to the nearest minute) that a customer must wait in a check-out line.

X	1	2	3	4	5
$P(X)$	0.1	0.32	0.4	0.15	0.03

What is the expected wait for a customer?

#5 In an essay titled "The Wager" by most compilers, Blaise Pascal employed the concept of expected value in an argument for faith in God. Pascal reasoned that the state of the world was such that God exists or that God does not exist and that every person has a choice to believe in God or not to believe. Pascal argued that man should believe in God even if the probabilities are very slight since the reward was infinite. Suppose life is a game for believers and that the probability of God's existence is only one in one million. Use an estimated value of $1,000,000.00 for a lifetime of tithing as a believer's loss if God does not exist. Use an estimated value of $\$\infty$ for a person's soul as a believer's gain if God does exist. What is the expected value associated with the game of life?