Lecture 1.3

College Algebra

Instruction: *Domain*

In a function, the independent variable represents the values of the first set of components called the domain. In practical terms, the possible values of the independent variable, usually denoted by x, represent the domain.

Many functions have a domain of all real numbers denoted in set interval notation as $(-\infty,\infty)$, meaning from negative infinity to positive infinity. Three important functions that we will study do not have a domain of all real numbers. Rational functions, that is functions with a variable in the denominator of a rational expression like $r(x)$ below, often have restrictions on their domain.

$$r(x) = \frac{x+1}{x-2}$$

A rational expression with zero as the divisor is undefined. Consequently, $r(x)$ has a limited domain since $r(x)$ cannot be defined when $x = 2$. Thus, the domain of $r(x)$ in set interval notation is $(-\infty,2)\cup(2,\infty)$. All rational functions have restrictions on their domain. These restrictions correspond to the x-values that render the denominator equal to zero. *To find the restrictions on a rational function, set the denominator equal to zero and solve.*

Similarly, functions with variables in the radicand of a radical with an even index like $R(x)$ below sometimes have limited domains:

index

$$R(x) = \sqrt[4]{x}$$

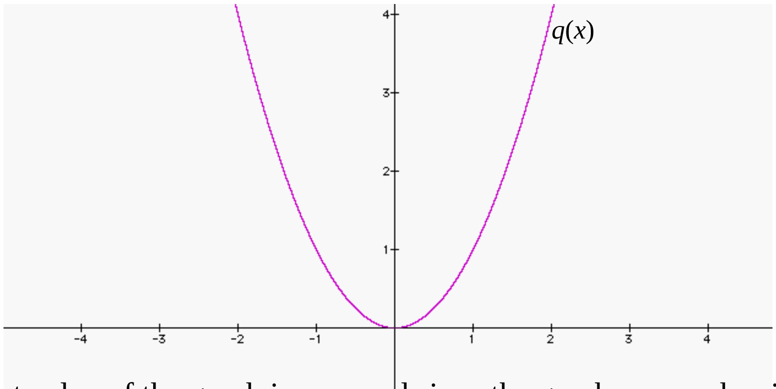
radicand

The radicand of even roots such as a square root or fourth root must be non-negative to render a real number result. Consequently, $R(x)$ has a limited domain since $R(x)$ does not exist as real numbers when $x < 0$. Thus, the domain of $R(x)$ in set interval notation is $[0,\infty)$. All functions with independent variables in the radicand of a radical with an even index have restrictions on their domain. These restrictions (x-values that are excluded) correspond to the x-values that render the radicand negative. *To find the domain (included* x*-values) of a function containing a radical with an even index, set the radicand greater than or equal to zero and solve.*

Logarithmic functions, which will be discussed later in the course, also have restricted domains.

Instruction: *Range*

In a function, the dependent variable represents the values of the second set of components called the range. In practical terms, the possible values of the dependent variable, usually denoted by y or $f(x)$, represent the range. Determining a function's range requires some understanding of the function's behavior. Consider $q(x) = x^2$. Since every x-value is squared to find the value of $q(x)$, the function never has a negative value. Thus, the range in set interval notation is $[0,\infty)$. The range can readily be ascertained from the function's graph.



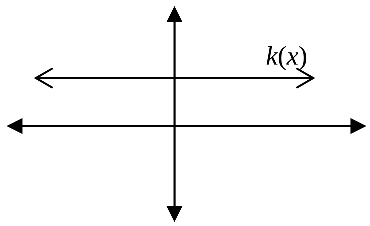
Since the lowest value of the graph is zero and since the graph approaches infinity on the *y*-axis without interruption, the range extends from zero to positive infinity $[0,\infty)$.

Instruction: ***Increasing/Decreasing/Constant Behavior***

A function increases along an interval if the function's values increase as the values of the independent variable increase. In other words, if the *y*-values increase as the *x*-values increase, the function increases. Consider the graph of $q(x)$ above. The function $q(x)$ increases as the *x*-values increase beginning with zero and extending forward. Thus, $q(x)$ increases along the interval $(0,\infty)$.

Likewise, a function decreases along an interval if the function's values decrease as the values of the independent variable increase. In other words, if the *y*-values decrease as the *x*-values increase, the function decreases. Consider the graph of $q(x)$ above. The function $q(x)$ decreases as the *x*-values increase from negative infinity to zero. Thus, $q(x)$ decreases along the interval $(-\infty,0)$.

If a function's value remains unchanged as the *x*-value changes, the function is constant. Consider $k(x)$ below.



Since the value of $k(x)$ never changes, it is a constant function.

An Informal Discussion continued. . .

Recall that a function is a special relation between two sets. A set is a collection of objects, and for our purposes those objects will be numbers and/or amounts. In this class, we will assume that the domain for our functions is the set of all real numbers noted as $\mathbb{R}$ or using set interval notation $(-\infty,\infty)$. There are some cases outlined below for which the domain will not include all real numbers.

I. The domain is stated as some subset of $\mathbb{R}$.

II. The operation of the function implies a domain restriction. Two such operations are division and taking even roots.

III. The word problem represented by the function implies a restriction.

In Sections 1.1 and 1.2, we discussed the monthly income of a paperboy who earns $4.50 for every subscriber to whom he delivers the paper. We used the function below to describe the paperboy's monthly income.

$$p(x) = \$4.50x$$

Since the "rule" of the function involves multiplication (multiply the domain value by $4.5 to get the corresponding range value), the operation of the function does not imply a domain restriction. In other words, since we can multiply any number by 4.5, the domain could be the set of real numbers, $\mathbb{R}$. Our function, however, represents a word problem where *x* (the domain variable) represents the number of subscribers. For our word problem, it does not make sense to assume the paperboy will deliver papers to a negative number of subscribers. In fact, it does not make a lot of sense to even include fractions since the paperboy probably will not have fractional number of subscribers. If we assume that the paperboy can deliver a maximum of 1,000 papers, we might use descriptive notation and say:

D = {a whole number of subscribers from 0 to 1,000}.

Using set-builder notation where $\mathbb{Z}$ indicates the set of integers, we have:

$$D = \{x \mid 0 \le x \le 1{,}000,\ x \in \mathbb{Z}\}.$$

Since the range of our function is the product of $4.5 and each element from the domain, we have:

$$R = \{y \mid \$0, \$4.5, \$9, \$13.5, \$18, \ldots \$4{,}500\}$$

Note that writing the range as a sequence of numbers reveals that $p(x)$ is an increasing function.

Instruction: ***Domain, Range, Behavior***

Example 1
Stating the Domain of a Function

Given $f(x) = 5 + \sqrt{-1 - x}$, what is the domain of $f(x)$?

The domain is understood to be all real numbers unless the operation implies a restriction. This function implies a restriction because taking the even root of a number only produces a real number result if the number is a non-negative number. To find the domain, find where the radicand is non-negative.

$$-1 - x \ge 0$$
$$-1 - x + x \ge 0 + x$$
$$-1 \ge x$$
$$x \le -1$$

The *x*-values must be less than or equal to negative one. Using set interval notation, the domain is $(-\infty, -1]$. The bracket next to negative one indicates that negative one is part of the domain.

Example 2
Stating the Domain of a Function

Given $Q(x) = \dfrac{x+1}{x^2 - 6x - 16}$, what is the domain of $Q(x)$?

The domain is understood to be all real numbers unless the operation implies a restriction. $Q(x)$ implies a restriction because division by zero is undefined. To find the domain, find where the denominator does not equal zero.

$$x^2 - 6x - 16 \ne 0$$
$$(x+2)(x-8) \ne 0$$
$$x + 2 \ne 0, \quad x - 8 \ne 0$$
$$x \ne -2, \qquad x \ne 8$$

The *x*-values may take any value except –2 and 8. The domain has three intervals of values: all the numbers less than –2, the numbers between –2 and 8, and all the numbers greater than 8; thus, $D = (-\infty, -2) \cup (-2, 8) \cup (8, \infty)$.

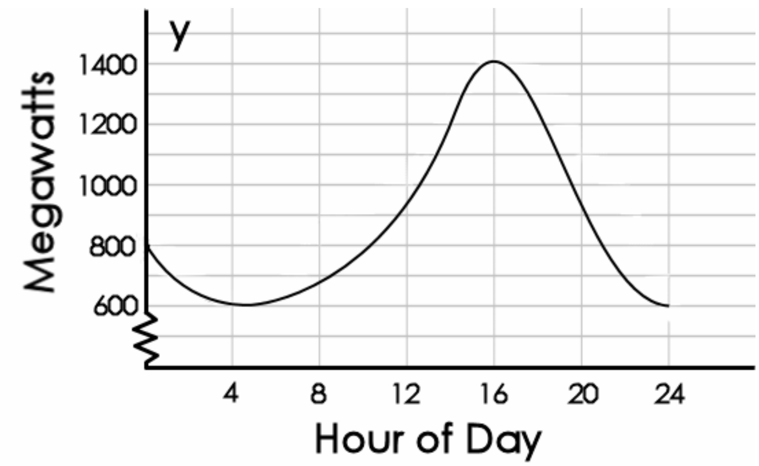
Example 3
Stating the Domain of a Function

Given $s(x) = x^2 + 2x + 1$, what is the domain of $s(x)$?

The domain is understood to be all real numbers unless the operation implies a restriction. The operations involved with s(x) do not imply a restriction. Any number can be multiplied by itself. Any number can be doubled, and any two numbers can be added together and added to one. The domain of $s(x)$ includes all real numbers denoted by $\mathbb{R}$ or written in set interval notation as $(-\infty, \infty)$.

Example 4
Finding the Range of a Function

What is the range of the function graphed below?



The range is the set of values assigned to correspond with the domain values. The vertical axis represents the range values. Identifying the range involves finding the lowest value and the highest value of the function and noticing any breaks in between. The graph above shows a function that is continuous--that is, it does not have any breaks--along a domain of [0,24]. The lowest value of 600 megawatts occurs during the fourth hour and twenty-fourth hour. The highest value of 1,400 megawatts occurs during the sixteenth hour. The range extends from 600 to 1,400. Using set interval notation, the range is $[600, 1400]$. The brackets indicate that 600 and 1,400 are both included in the range.

Example 5
Recognizing Behavior

Given $F(x) = (x - 3)^4$, for what value of x does the $F(x)$ change behavior?

Behavior refers to whether or not the function's values decrease, remain constant, or increase as x-values increase. Examining a table of values of the function for increasing x-values helps determine the behavior. Arbitrarily pick some x-value, below negative eight is chosen, and then increasing x-values. Look for a change in behavior.

x	$(x - 3)^4$	$F(x)$
–8	$(-8-3)^4$	14,641
–6	$(-6-3)^4$	6,561
–4	$(-4-3)^4$	2,401
–2	$(-2-3)^4$	625
0	$(0-3)^4$	81
2	$(2-3)^4$	1
4	$(4-3)^4$	1

The function decreased until two, then a change seemed to have occurred. Choosing x-values closer together may help determine what happens between two and four.

x	$(x - 3)^4$	$F(x)$
2	$(2-3)^4$	1
2.5	$(2.5-3)^4$	0.0625
3	$(3-3)^4$	0
3.5	$(3.5-3)^4$	0.0625
4	$(4-3)^4$	1

This exercise helps make it clear that as x-values increased from negative numbers (with large absolute values) to x-values close to three, the function decreases, but the function starts increasing thereafter.

The function changes behavior from decreasing to increasing at $x = 3$.

Practice Set 1.3A

State the domain of each of the following functions.

#1 $a(x) = \sqrt{x+3}$

#2 $b(x) = \dfrac{x}{x+1}$

#3 $c(x) = 6x^5 + 3x^2 + 1$

#4 $d(x) = \sqrt{x}$

#5 $f(x) = \dfrac{x}{x-8}$

#6 $g(x) = \dfrac{x^2 - x - 12}{x^2 - x - 56}$

#7 $h(x) = \sqrt{10 - x}$

#8 $j(x) = \dfrac{x^2 - 4}{14 - x}$

#9 $k(x) = 5x^4 + 3x^3 - 6x^2 + 4x - 11$

#10 $p(x) = \pi x - 7$

#11 $n(x) = \sqrt[3]{17 - x}$

#12 $r(x) = \dfrac{x^2 - x}{25 - x^2}$

#13 $M(x) = \sqrt{\dfrac{x}{x+3}}$

#14 $n(x) = \sqrt[3]{\dfrac{7}{x^2 + 7}}$

#1 [-3,∞), #2 (-∞,-1)U(-1, ∞), #3 (-∞,∞), #4 [0, ∞), #5 (-∞,8)U(8, ∞), #6 (-∞,-7)U(-7,8)U(8, ∞), #7 (-∞,10], #8 (-∞,14)U(14, ∞), #9 (-∞,∞), #10 (-∞,∞), #11 (-∞,∞), #12 (-∞,-5)U(-5,5)U(5, ∞), #13 (-∞,-3)U[0,∞), #14 (-∞,∞)

Practice Set 1.3A_Supplemental

State the domain of each of the following functions.

#1 $a(x)=\sqrt{5-x}$

#2 $P(x)=x^4+7x+12$

#3 $R(x)=\dfrac{x^4+7x^2+12}{6x^2+13x+6}$

#4 $r(x)=\sqrt{x^2}$

#5 $Q(x)=\dfrac{x}{5x-4}$

#6 $q(x)=\dfrac{50-x}{x-7}$

#7 $h(x)=\sqrt{1-x}$

#8 $H(x)=\sqrt[5]{100x}$

#9 $p(x)=14x^5-5x^4+13x^3-22x^2+9x-12$

#10 $n(x)=\sqrt[4]{17-x}$

#11 $L(x)=\dfrac{4}{5}x+2$

#12 $r(x)=\dfrac{6}{4x^2-1}$

#13 $M(x)=\sqrt{\dfrac{1}{x}}$

#14 $G(x)=\dfrac{x}{4x-3}$

#1 $(-\infty,5]$
#2 $(-\infty,\infty)$
#3 $(-\infty,-\frac{3}{2})\cup(-\frac{3}{2},-\frac{2}{3})\cup(-\frac{2}{3},\infty)$
#4 $(-\infty,\infty)$
#5 $(-\infty,\frac{4}{5})\cup(\frac{4}{5},\infty)$
#6 $(-\infty,7)\cup(7,\infty)$
#7 $(-\infty,1]$
#8 $(-\infty,\infty)$
#9 $(-\infty,\infty)$
#10 $(-\infty,17]$
#11 $(-\infty,\infty)$
#12 $(-\infty,-\frac{1}{2})\cup(-\frac{1}{2},\frac{1}{2})\cup(\frac{1}{2},\infty)$
#13 $(0,\infty)$
#14 $(-\infty,\frac{3}{4})\cup(\frac{3}{4},\infty)$

Practice Set 1.3B

Use tables of values or graphs of the following functions to determine their intervals of increasing behavior.

#1 $a(x) = \sqrt{x+5}$

#2 $b(x) = \dfrac{x}{x+1}$

#3 $c(x) = x^4 - 8x^2 + 16$

#4 $f(x) = \dfrac{x}{x-8}$

Use tables of values or graphs of the following functions to determine their intervals of decreasing behavior.

#5 $h(x) = \sqrt{2-x}$

#6 $p(x) = -\pi x$

#7 $P(x) = \pi x$

#8 $f(x) = \dfrac{-4}{2-x}$

Use graphs of the following functions to determine their range.

#9 $q(x) = x^2 + 1$

#10 $R(x) = \sqrt{x+1}$

#1 (-5,∞), #2 (-∞,-1)U(-1, ∞) #3 (-2,0)U(2,∞), #4 {}, #5 (-∞,2), #6 (-∞,∞), #7 { }, #8 (-∞,2)U(2,∞), #9 [1, ∞), #10 [0, ∞)

Practice Set 1.3B_Supplemental

Use tables of values or graphs of the following functions to determine their range and intervals of increasing, decreasing, or constant behavior.

#1 $T(x) = \sqrt{8x-1}$

#2 $U(x) = \frac{10}{x+2}$

#3 $V(x) = -\frac{6x+5}{1-2x}$

HINT for #3: $V(x)$ has one range restriction. To find the range restriction, evaluate the function as x-values increase to large values like 1,000 & 10,000. Do the same as x-values decrease to small values like -1,000 and -10,000.

#4 $W(x) = 2x^2 - 5x - 3$

HINT for #4: Finding the vertex of the parabola will help to establish the range as well as the intervals of increasing and decreasing behavior.

CHALLENGE PROBLEM:

#5 $Y(x) = \frac{x^2}{x-1}$

#1 range: $[0,\infty)$
increasing behavior: increases on the interval $(\frac{1}{8},\infty)$, i.e., it increases throughout domain, which is $[\frac{1}{8},\infty)$.

#2 range: $(-\infty,0)U(0,\infty)$
decreasing behavior: decreases throughout domain of $(-\infty,-2)U(-2,\infty)$

#3 range: $(-\infty,3)U(3,\infty)$;
decreasing behavior: decreases throughout domain of $(-\infty,\frac{1}{2})U(\frac{1}{2},\infty)$

#4 range: $[-6.125, \infty)$;
decreasing behavior: $(-\infty,1.25)$;
increasing behavior: $(1.25,\infty)$

#5 range: $(-\infty,0]U[4,\infty)$;
increasing behavior: $(-\infty,0)U(2,\infty)$;
decreasing behavior: $(0,1)U(1,2)$

Assignment 1.3

Problems

State the domain of the following functions.

1 $y(x) = x + 7\pi$

2 $r(x) = \sqrt[3]{18 - x}$

3 $Q(x) = \dfrac{2x}{x+5}$

4 $g(x) = \dfrac{x+2}{x^2 - x - 20}$

5 $p(x) = 17x^3 + 3x^2 - \pi x$

6 $R(x) = \sqrt{20 - x}$

7 $f(x) = x^{2/3} - 1$

8 $q(x) = \dfrac{4}{x^2 + 2}$

$$h(x) = \sqrt{x+3}$$

Answer the following questions. It will be helpful to generate a t-table of values and to graph the function.

9 What is the domain of $h(x)$?

10 What is the range of $h(x)$?

11 Along what intervals is $h(x)$ increasing?

12 Along what intervals is $h(x)$ decreasing?

$$d(t) = t^2 + 1$$

Answer the following questions. It will be helpful to generate a t-table of values and to graph the function.

13 What is the domain of $d(t)$?

14 What is the range of $d(t)$?

15 Along what intervals is $d(t)$ increasing?

16 Along what intervals is $d(t)$ decreasing?

College Algebra

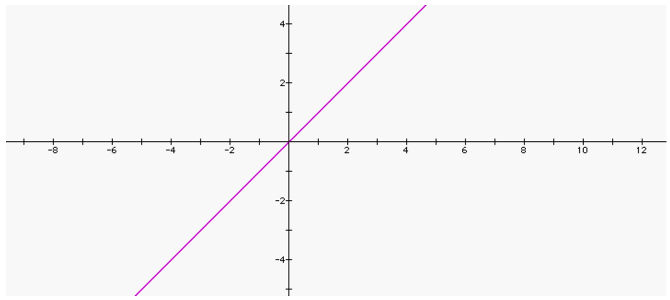
Instruction: *Six Elementary Functions*

This lecture discusses six elementary functions. The serious student would memorize these six functions.

$$\text{I.}\quad f(x) = x$$

As defined in *Box I*, $f(x)$ is a linear function. The function increases throughout its domain, which includes all real numbers. The range also includes all real numbers.

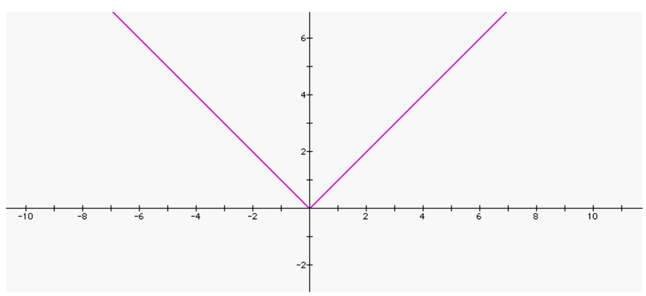
x	$f(x)$
–2	–2
–1	–1
0	0
1	1
2	2



$$\text{II.}\quad g(x) = |x|$$

As defined in *Box II*, $g(x)$ is an absolute value function. It decreases from negative infinity to zero $(-\infty, 0)$ and increases thereafter $(0, \infty)$. The function is symmetrical about the y-axis. Consequently, the function is an even function where $g(x) = g(-x)$. The minimum value of $g(x)$ is zero, so the range is $(0, \infty)$.

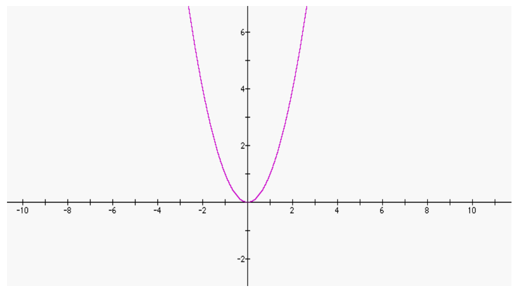
x	$g(x)$
–2	2
–1	1
0	0
1	1
2	2



III. $h(x) = x^2$

As defined in *Box III*, $h(x)$ is a quadratic function with a vertex at the origin (0,0). The function decreases from negative infinity to zero $(-\infty, 0)$ and increases thereafter $(0, \infty)$. Thus, the *y*-value of the vertex represents the absolute minimum value of the function, and the range is $(0, \infty)$. The function is symmetrical about the *y*-axis. Consequently, the function is an even function where $h(x) =$ $h(-x)$.

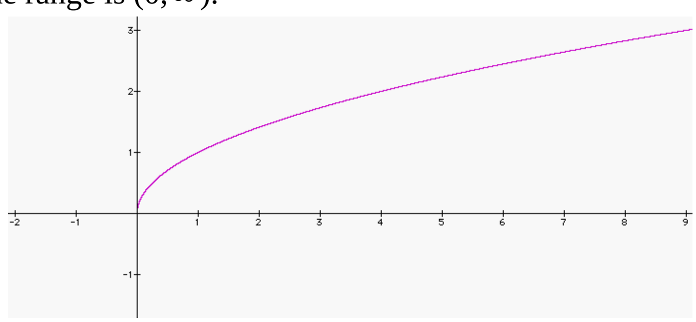
x	$h(x)$
–2	4
–1	1
0	0
1	1
2	4



IV. $j(x) = \sqrt{x}$

As defined in *Box IV*, $j(x)$ is a square root function. The domain of $j(x)$ is limited to where the radicand is positive; thus, the domain is where $x \geq 0$. The function increases throughout its domain. Its minimum value is zero, so the range is $(0, \infty)$.

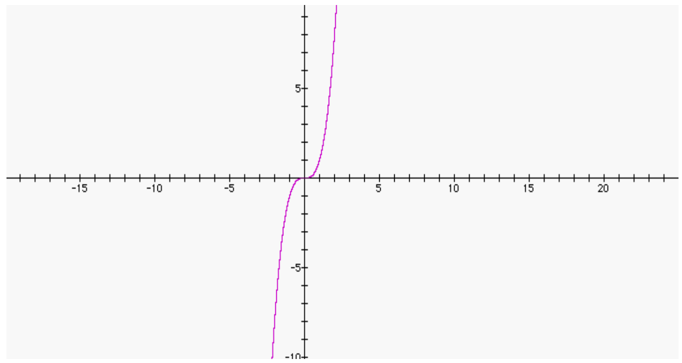
x	$j(x)$
–1	non-real number
0	0
1	1
2	$\sqrt{2}$
4	2



V. $k(x) = x^3$

As defined in *Box V*, $k(x)$ is a cubic function. This function is symmetrical about the origin, so it is an odd function where $-k(x) = k(-x)$. The domain and range both include all real numbers.

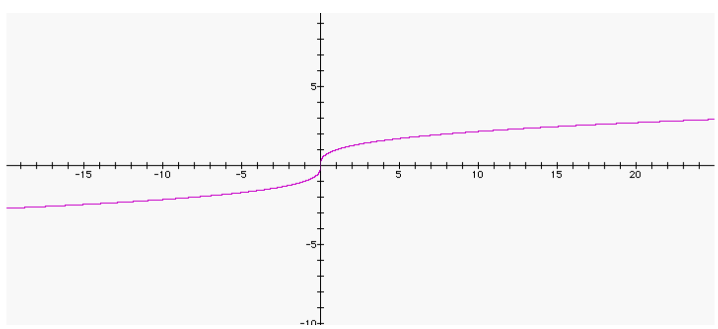
x	$k(x)$
–2	–8
–1	–1
0	0
1	1
2	8



VI. $q(x) = \sqrt[3]{x}$

As defined in *Box VI*, $q(x)$ is a cube-root function. This function is symmetrical about the origin, so it is an odd function where $-q(x) = q(-x)$. The domain and range both include all real numbers.

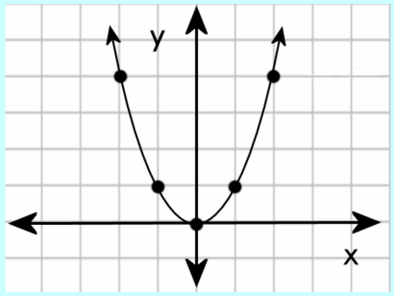
x	$q(x)$
–8	–2
–2	$-\sqrt[3]{2}$
–1	–1
0	0
1	1
2	$\sqrt[3]{2}$
8	2



Instruction: ***Elementary Functions***

Example 1
Elementary Functions

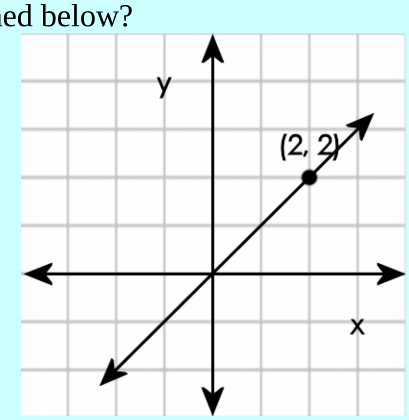
Name the function graphed below.



The function is a quadratic function, $y(x) = x^2$.

Example 2
Elementary Functions

Name the function graphed below?



The function is the linear function $y(x) = x$.

Example 3
Elementary Functions

State the domain of $y(x) = \sqrt{x}$.

The radicand of a square root must be a non-negative value. The domain of the square root function includes non-negative numbers: $[0, \infty)$.

Example 4
Elementary Functions

State the range of $y(x) = |x|$.

The absolute value of a non-zero number is positive. The absolute value of zero is zero. The range of the absolute value function is $[0, \infty)$.

Example 5
Elementary Functions

Consider the cubic function $y(x) = x^3$. Is the function an increasing or decreasing function?

The cubes of successively larger numbers are themselves successively larger. As *x*-values increase, the *y*-values increase. The function increases.

Example 6
Elementary Functions

Consider the cube-root function $y(x) = \sqrt[3]{x}$. Is the function an increasing or decreasing function?

The cubed-roots of successively larger numbers are themselves successively larger. As *x*-values increase, the *y*-values increase. The function increases.

Practice Set 1.4

Problems

Label each of the functions below as linear, absolute value, quadratic, square-root, cubic, or cube-root functions. Describe the transformation applied to the elementary function (described in Section 1.7) of the same name.

#1 $C(x) = x^{\frac{1}{3}} + 1.5$

Hint: $a^{\frac{s}{r}} = \sqrt[r]{a^s}$.

#2 $s(t) = -t^2$

#3 $T(x) = \sqrt{-x}$

#4 $V(x) = \dfrac{|x+7|}{2}$

#5 $p(x) = x^2 - 10x + 25$

Hint: The trinomial factors.

#6 $A(t) = -17 + x$

#7 $L(x) = x^3 + \pi$

.

#8 $F(x) = 5x$

#1 cube-root function, shifted up 1.5 units
#2 quadratic function, reflected over the x-axis
#3 square-root function, reflected over the y-axis
#4 absolute value function, $V(x) = \frac{1}{2}|x+7|$, shifted left seven units and dilated by a factor of ½
#5 quadratic function, $p(x) = (x+5)^2$, shifted right 5 units
#6 linear function, shifted down 17 units
#7 cubic function shifted up pi units
#8 linear function, dilated by a factor of 5

Assignment 1.4

Problems

Label each of the functions below as linear, absolute value, quadratic, square-root, cubic, or cube-root functions.

#1 $y(x) = x^2 + 17$

#2 $z(t) = -t$

#3 $b(x) = x^3 - 1$

#4 $\beta(x) = \dfrac{\sqrt{x+4}}{2}$

#5 $P(x) = x^2 + 12x + 36$

#6 $A(t) = |t+1|$

#7 $s(t) = x^{\frac{1}{2}} + 1$

Hint: $a^{\frac{1}{2}} = \sqrt{a}$.

#8 $T(x) = x^{\frac{1}{3}} - 1$

Hint: $a^{\frac{s}{r}} = \sqrt[r]{a^s}$.