College Algebra

Instruction: *Definition of Functions*

The BIG IDEA in College Algebra is the idea of a "function." A function is a rule that produces a correspondence between two sets of elements such that to each element in the first set there corresponds *one and only one* element in the second set.

Consequently, a relation between a set of first components (called the domain) and a set of second components (called the range) can be readily identified as a functional relation if each element in the first set of components does not correspond to more than one element in the second set.

Alternately defined, a function is a rule that pairs each element x from a set called the domain with *one and only one* element y from a set called the range.

In some situations, it is more convenient to think of functions as sets of ordered pairs. In other situations, functions are best viewed as rules, which are stated in the form of equations.

Functions relate two quantities, the elements of two sets, domain and range. These quantities are called variables because they can vary or change. The variable from the domain is the independent or input variable while the variable from the range is the dependent or output variable. Commonly, the independent variable is represented by x, and the dependent variable is represented by y.

The following table of ordered pairs represents a set that defines a function since no two ordered pairs have the same first component and different second components.

x	y
–2	4
–1	1
0	0
1	1
2	4

The next table of ordered pairs, however, does not represent a functional set since it contains ordered pairs with the same first component and different second components.

x	y
4	–2
1	–1
0	0
1	1
4	2

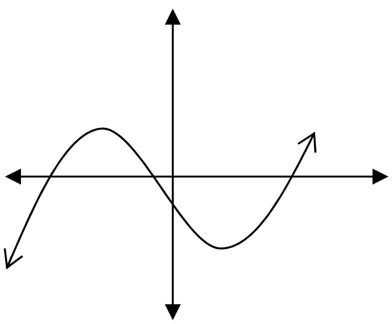
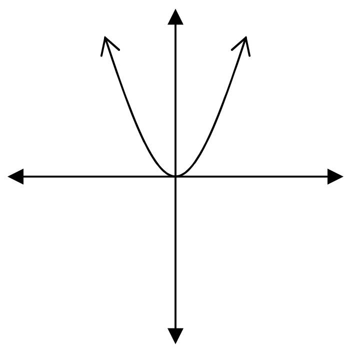
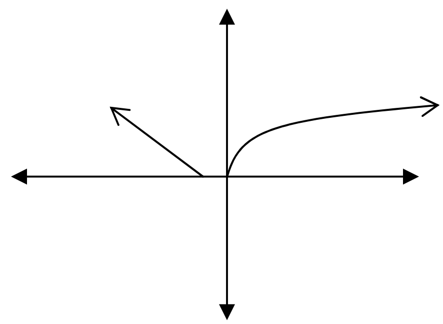
Lecture 1.1

In terms of equations, a function is an equation that generates a set of solutions or ordered pairs with the property that no two ordered pairs have the same first component and different second components.

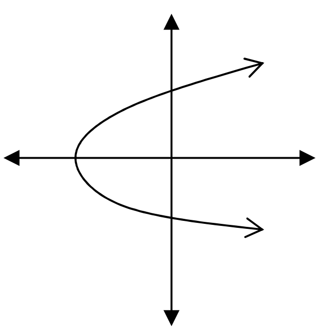
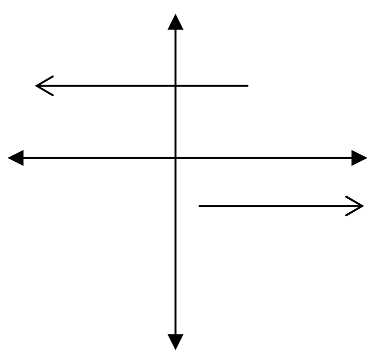
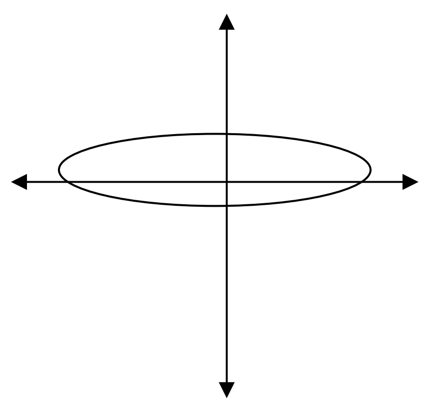
The equation $y + x^2 = 5$ represents a function because every x-value has only one corresponding y-value.

The equation $y^2 + x^2 = 5$ does not represent a function because some x-values have two corresponding y-values. For instance, if $x = 1$, $y = 2$ or $y = -2$.

Relations graphed on a Cartesian plane (an x-y plane) can be easily identified as functions using the vertical line test. If all vertical lines on the Cartesian plane will intersect the relation no more than once, then the relation is a function. Consequently, the following relations are functions.

The following relations are not functions since it is possible to intersect each more than once with a vertical line.

An Informal Discussion . . .

Let's think of a paperboy who gets paid per subscriber. Let's assume that the paperboy delivers papers to x number of subscribers. If the newspaper pays the paperboy \$4.50 per month per subscriber, we can write a monthly income function for the paperboy:

$$y = \$4.50x$$

The "rule" says multiply the input (or independent) variable by 45 to get the output (or dependent) variable. The input (independent) variable is the number of subscribers. The monthly income is the output of the function (or dependent variable). For example, if the paperboy has 100 subscribers, he earns \$450.00 per month, i.e., if $x = 100$, $y = \$450.00$.

The input/independent (x) variable represents a set of values called the domain.

The output/dependent (y) variable represents a set of values called the range.

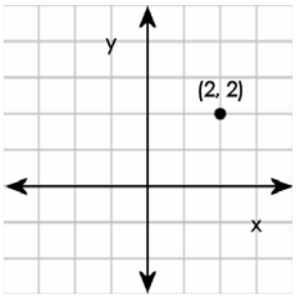
For the relation to be a function, there can be only one range value for any given domain value, and we see that this is the case with the paperboy function. If he has 100 subscribers, he gets paid \$450.00 not any other amount.

Instruction: *Definition of Functions*

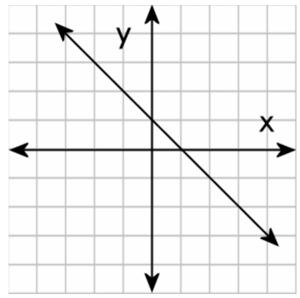
Example 1
Identifying the Graph of a Function

Which of the graphs below are graphs of functions?

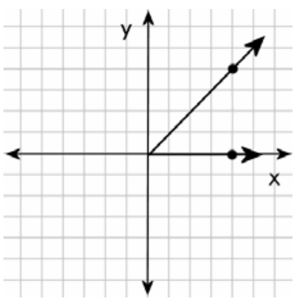
I.



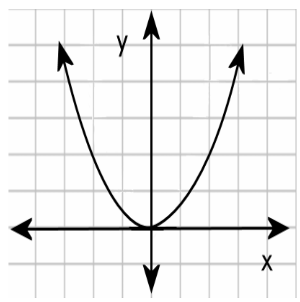
II.



III.



IV.



Graphs I., II., and IV. are graphs of functions because any vertical line imposed over the graphs will only intersect the graph in at most one point. Graph III., however, is not the graph of a function since a vertical line will intersect the graph more than once. Consider the vertical line $x = 4$, which will intersect the graph at (4,0) and (4,4).

Example 2
Determining Whether an Equation is a Function

Does the equation $x + y^2 = 4$ represent a function?

Since the question does not state which variable represents the domain (input) values and which represents the range (output) values, the question is ambiguous. This course assumes, however, that *y* will always represent the range, which is traditional. Solving the equation for *y*, will help determine if the equation represents a function.

$$x + y^2 = 4$$
$$y^2 = 4 - x$$
$$\sqrt{y^2} = \pm\sqrt{4 - x}$$
$$y = \pm\sqrt{4 - x}$$

Remember to use the $\pm$ symbol whenever taking the square root of both sides of an equation.

For particular values of *x*, e.g., *x*-values between –4 and 4, the equation is true for more than one *y*-value. For instance, substituting 3 for *x*, the equation yields the following result.

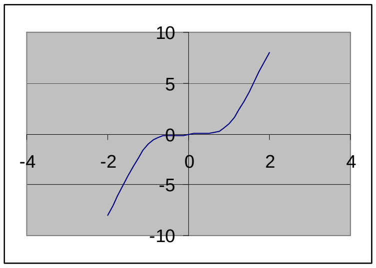
$$y = \pm\sqrt{4 - 3}$$
$$y = \pm\sqrt{1}$$
$$y = \pm 1$$

The equation represents a rule that assigns two numbers, 1 and –1, to correspond with the *x*-value 3; therefore, the equation does *not* represent a function since a function will only assign at most *one and only one y*-value to any given *x*-value.

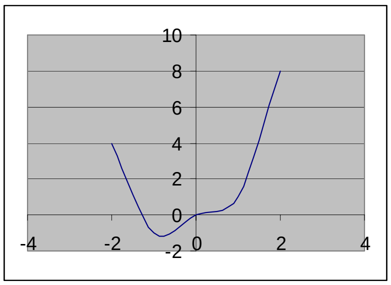
Practice Set 1.1

Instructions: Decide if each relation is a function. If the relation is a function, state why. If it is not, state why not.

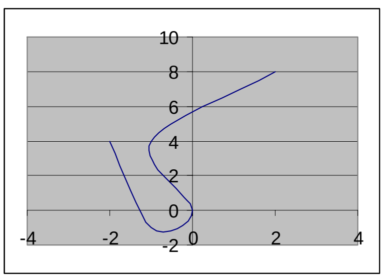
#1



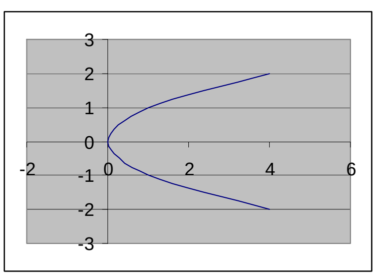
#2



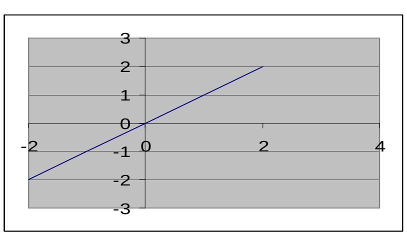
#3



#4



#5



#6

x	*y*
1	7
2	9
3	11
4	13
5	15

#7

x	*y*
-5	5
-1	1
0	0
1	1
5	5

#8

x	*y*
-1	2, -2
0	0
1	2, -2

#9 $5x + 7y = 32$

#10 $x^2 + y^2 = 25$

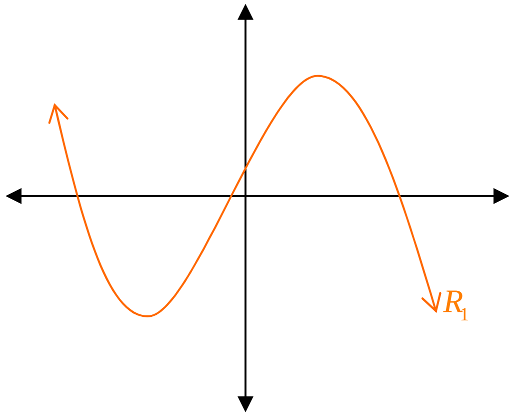
#1 function/passes vertical line test, #2 function/passes vertical line test, #3 not a function/fails vertical line test, #4 not a function/fails vertical line test, #5 function/passes vertical line test, #6 function because every *x*-value has only one corresponding *y*-value, #7 function because every *x*-value has only one corresponding *y*-value, #8 not a function because some of the *x*-values have more than one corresponding *y*-value, #9 function (students should recognize this as a linear function written in standard form) because every *x*-value has only one corresponding *y*-value, #10 not a function (students might recognize this as an equation of a circle) because *x*-values have more than one corresponding *y*-value, for instance both ordered pairs (4,3) and (4,-3) are solutions to the equation

Study Exercise 1.1

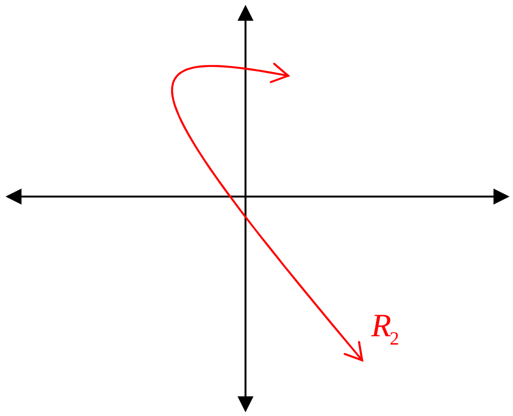
Problems

Use the vertical line test to determine whether the following relations are functions. State the answer with a sentence on the line provided.

#1



#2



Lecture 1.2

College Algebra

Instruction: *Functional Notation*

$f(x) = x - 4$

Functional notation replaces the dependent variable with the symbol $f(x)$ such that the ordered pair $(x, f(x))$ belongs to the function f. The symbol $f(x)$, therefore, represents the real number in the range of f (where range is the set of values of the dependent variable) corresponding to the domain value x (where domain is the set of values of the independent variable). Consequently, the notation $f(4) = 0$ represents the ordered pair (4,0). Evaluating $f(2)$ requires the substitution of 2 in for the independent variable:

$$f(x) = x - 4$$
$$f(2) = 2 - 4$$
$$f(2) = -2$$
$$(2, -2)$$

If x is a real number that is not in the domain of f, then f is not defined at x and $f(x)$ does not exist.

Any symbol can be similarly substituted for f so that one can discuss multiple functions such as f and g.

An Informal Discussion continued. . .

In Section 1.1, we discussed the monthly income of a paperboy who earns $4.50 for every subscriber to whom he delivers the paper. We used the equation below to describe the paperboy's monthly income.

$$y = \$4.50x$$

For functions like this, we use a type of notation called function notation where $f(x)$ represents y (and the "f" is arbitrary and can be any letter or symbol).

For example, we might write $p(x) = \$4.50x$ instead of $y = \$4.50x$. This notation gives us a shorthand where $p(100)$ stands for the function's value (that is, the function's y-value) when $x = 100$ so that $p(100) = \$450.00$.

Evaluating $p(50)$ equates to finding the function's value (the range value) when $x = 50$ as shown below.

$$p(50) = \$4.5(50)$$
$$p(50) = \$225.00$$

So, the paperboy earns $225.00 per month when he has 50 subscribers. :)

Instruction: *Functional Notation*

Example 1
Evaluating Functions

Given $f(x) = \sqrt[3]{x}$ and $g(x) = \sqrt{2x}$, which is greater $f(125)$ or $g(50)$?

To evaluate $f(125)$, substitute 125 for *x* in the equation for *f*.

$$f(x) = \sqrt[3]{x}$$
$$f(125) = \sqrt[3]{125} = 5$$

To evaluate $g(50)$, substitute 50 for *x* in the equation for *g*.

$$g(x) = \sqrt{2x}$$
$$g(50) = \sqrt{2 \cdot 50} = \sqrt{100} = 10$$

Clearly, 10 is greater than 5; thus, $g(50) > f(125)$.

Example 2
Evaluating Functions

Given $q(x) = x^2 - 2$, which of the following is negative?

$$q(8),\ q(-5),\ q\left(\frac{2}{3}\right),\ q(0),\ q(-2)$$

To evaluate $q(8)$, substitute 8 for *x* in the equation for *q*.

$$q(x) = x^2 - 2$$
$$q(8) = (8)^2 - 2 = 64 - 2 = 62$$

To evaluate $q(-5)$, substitute -5 for *x* in the equation for *q*.

$$q(x) = x^2 - 2$$
$$q(-5) = (-5)^2 - 2 = 25 - 2 = 23$$

To evaluate $q\left(\frac{2}{3}\right)$, substitute 2/3 for *x* in the equation for *q*.

$$q(x) = x^2 - 2$$
$$q\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 - 2 = \frac{4}{9} - 2 = -\frac{14}{9}$$

To evaluate $q(0)$, substitute 0 for *x* in the equation for *q*.

$$q(x) = x^2 - 2$$
$$q(0) = (0)^2 - 2 = -2$$

To evaluate $q(-2)$, substitute -2 for *x* in the equation for *q*.

$$q(x) = x^2 - 2$$
$$q(-2) = (-2)^2 - 2 = 4 - 2 = 2$$

Only $q\left(\frac{2}{3}\right)$ and $q(0)$ are negative.

Example 3
Using Function Notation

Given $f(x) = 5 - x^2$, simplify the expression $\dfrac{f(x+h) - f(x)}{h}$.

Note that the numerator of the expression is a difference of *f*-values. Evaluate $f(x+h)$.

$$f(x) = 5 - x^2$$
$$f(x+h) = 5 - (x+h)^2$$
$$f(x+h) = 5 - \left[(x+h)(x+h)\right]$$
$$f(x+h) = 5 - \left[x^2 + xh + hx + h^2\right]$$
$$f(x+h) = 5 - \left[x^2 + 2xh + h^2\right]$$
$$f(x+h) = 5 - x^2 - 2xh - h^2$$

Find the difference of $f(x+h)$ and $f(x)$.

$$\begin{aligned} f(x+h) - f(x) &= 5 - x^2 - 2xh - h^2 - (5 - x^2) \\ &= 5 - x^2 - 2xh - h^2 - 5 + x^2 \\ &= -2xh - h^2 \end{aligned}$$

Simplify the ratio $\dfrac{f(x+h) - f(x)}{h}$.

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{-2xh - h^2}{h} \\ &= \frac{-2x\cancel{h}}{\cancel{h}} - \frac{h^{\cancel{2}}}{h} \\ &= -2x - h \end{aligned}$$

Practice Set 1.2

Given the functions: $f(x)=\sqrt{22-x}$, $g(x)=-5x+1$, $j(x)=x^2+1$, evaluate the following expressions.

#1 $f(6)$

#2 $g(0)$

#3 $f(-3)$

#4 $j(-1)$

#5 $f(-27)$

#6 $g(6)$

#7 $j(0)$

#8 $f(22)$

#9 $j(12)$

#10 $g(-2)$

#11 $f(26)$

#12 $f(20)$

#13 $j(\theta)$

#14 $g(\lambda)$

#15 $g(-1)$

#16 $g(a+2)$

#17 $f(\sqrt{2})$

#18 $j(a)$

#19 $g(j(x))$

#20 $j(g(x))$

#21 $g(a-1)$

#22 $g(a+h)$

#23 $g(a)+g(2)$

#24 $j(a+h)$

#25 $j(a+h)-j(a)$

#26 $\dfrac{j(a+h)-j(a)}{h}$

#1 4, #2 1, #3 5, #4 2, #5 7, #6 –29, #7 1, #8 0, #9 145, #10 11, #11 not a real number, #12 $\sqrt{2}$, #13 θ^2+1, #14 $-5\lambda+1$, #15 6, #16 $-5a-9$, #17 $\sqrt{22-\sqrt{2}}$, #18 a^2+1, #19 $-5x^2-4$, #20 $25x^2-10x+2$, #21 $-5a+6$, #22 $-5a-5h+1$, #23 $-5a-8$, #24 $a^2+2ah+h^2+1$, #25 h^2+2ah, #26 $2a+h$

Study Exercise 1.2

Problems

Consider functions *f* and *g* given below.

$$f(x) = 5x^2 + 1 \qquad\qquad g(x) = 7 - x$$

Evaluate the following expressions.

#1 $g(-1)$

#2 $g(1)$

#3 $f(3)$

#4 $f(a+h)$