

Instruction: Operations with Matrices

Two matrices are said to be equal if they have the same size and their corresponding elements are *equal*. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & (2+2) \end{bmatrix}$$

If all the elements are not equal then the matrices are not equal.

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$$

A zero matrix is a matrix in which all elements are zero. The zero matrix is commonly denoted O .

Two matrices A and B of the same size can be added or subtracted to produce a matrix of the same size. This is done by adding or subtracting the corresponding elements in the two matrices. For example,

$$\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 3 & -4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1-0 & 2-2 \\ -3-3 & 1-(-4) \\ 0-2 & 5-(-1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 3 & -4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -6 & 5 \\ -2 & 6 \end{bmatrix}$$

The additive commutative property, additive associate property, and the additive identity property all hold for matrix addition. Accordingly,

$$A + B = B + A$$

$$(A + B) + C = A + (B + C)$$

$$A + O = O + A = A$$

A matrix A may be multiplied by a real number called a *scalar* in the context of matrix algebra. The scalar product, denoted by cA , is a matrix obtained by multiplying each entry of A by c . For example,

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$\text{Then } 3A = 3 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 0 & 3 \cdot 1 \end{bmatrix}$$

$$3A = 3 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 0 & 3 \end{bmatrix}$$

With certain restrictions, one matrix may be multiplied by another matrix. The following problem will help define matrix multiplication.

On a certain day Maria's Optical sold 32 frames, 30 poly-carbonate lenses, and 2 no-line bifocal lenses. If the prices for these goods on that day were \$50 for frames, \$65 for poly-carbonate lenses, and \$45 for no-line bifocal lenses, find the total revenue for Maria's Optical for that day. The day's sale of goods may be represented by a matrix:

$$A = [32 \quad 30 \quad 2].$$

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The sale prices for that day may be represented by another matrix:

$$B = \begin{bmatrix} 50 \\ 65 \\ 45 \end{bmatrix}.$$

The first element in matrix A gives the number of frames sold and the first element in matrix B gives the selling price for each frame. Their product $32 \cdot 50$ yields the revenue from sold frames for the day. A similar interpretation of the second and third elements in the two matrices suggests that multiplying the corresponding elements will obtain the respective revenues gained from the sale of each product. Adding each of these sub-products would yield the total revenue obtained from the sale of goods that day. Thus, multiplying the elements of matrix A by the corresponding element in matrix B and adding the products yields a single number result representing the total revenue of the shop:

$$A = [32 \quad 30 \quad 2]$$

$$1 \times 3$$

$$B = \begin{bmatrix} 50 \\ 65 \\ 45 \end{bmatrix}$$

$$3 \times 1$$

$$[32 \quad 30 \quad 2] \cdot \begin{bmatrix} 50 \\ 65 \\ 45 \end{bmatrix} = 32 \cdot 50 + 30 \cdot 65 + 2 \cdot 45$$

$$[32 \quad 30 \quad 2] \cdot \begin{bmatrix} 50 \\ 65 \\ 45 \end{bmatrix} = 1,600 + 1,950 + 90$$

$$[32 \quad 30 \quad 2] \cdot \begin{bmatrix} 50 \\ 65 \\ 45 \end{bmatrix} = 3,640$$

$$AB = [3,640]$$

$$1 \times 1$$

The shop made \$3,640 from the sale of goods.

The above problem exemplifies a *matrix product*. In matrix multiplication each element of the matrix product is the sum of the products of the corresponding row from the first matrix and the corresponding column in the second matrix.

In general terms, if A is a matrix of dimension $m \times n$ and B is a matrix of dimension $n \times p$ (the number of columns of A equal the number of rows of B), then the matrix product of A and B , AB , is defined and is a matrix of dimension $m \times p$. Schematically,

$$\begin{array}{ccc} \text{Size of } A & & \text{Size of } B \\ m \times n & & n \times p \\ \wedge \text{-----} m \times p \text{-----} \wedge \\ & \text{Size of } AB & \end{array}$$

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In less general terms, the following example will illustrate the mechanics of matrix multiplication.

$$\text{If } A = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 & -3 \\ 4 & -1 & 2 \\ 2 & 4 & 1 \end{bmatrix}$$

Compute AB .

The dimension of matrix A is 2×3 , and the dimension of matrix B is 3×3 . Since the number of columns of matrix A is equal to the number of rows of matrix B , the matrix product, AB , is defined. (Note that the matrix product BA is not defined since the number of columns in matrix B does not equal the number of rows in matrix A . This phenomenon shows that the commutative property does not hold for matrix multiplication.) Furthermore, the dimension of matrix AB is 2×3 . Thus,

$$\begin{bmatrix} 3 & 1 & 4 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & -3 \\ 4 & -1 & 2 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

To determine the element c_{11} , multiply the elements of Row 1 of A by the corresponding elements of Column 1 of B and add the products. The position of each element indicates the procedure that obtains it. The element c_{11} occupies the first row and first column of the matrix product. Thus, it is obtained by multiplying the elements of the first row of A by the corresponding elements of the first column of B and adding the products.

$$c_{11} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = 3 \cdot 1 + 1 \cdot 4 + 4 \cdot 2 = 15$$

Similarly, element c_{12} occupies the position of the first row second column in the matrix product. Therefore, it is obtained by multiplying the elements of the first row of A by the corresponding elements of the second column of B and adding the products.

$$c_{12} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = 3 \cdot 3 + 1(-1) + 4 \cdot 4 = 24$$

The remaining elements are found accordingly.

$$c_{13} = \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 3(-3) + 1 \cdot 2 + 4 \cdot 1 = -3$$

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$$c_{21} = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = (-1)1 + 2 \cdot 4 + 3 \cdot 2 = 13$$

$$c_{22} = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = (-1)3 + 2(-1) + 3 \cdot 4 = 7$$

$$c_{23} = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = (-1)(-3) + 2 \cdot 2 + 3 \cdot 1 = 10$$

Thus, the matrix product AB is $\begin{bmatrix} 15 & 24 & -3 \\ 13 & 7 & 10 \end{bmatrix}$.

As mentioned above, in general, $AB \neq BA$ for matrices A and B , so the commutative property does not hold for matrix multiplication. However, the associative and distributive properties are valid. So, $(AB)C = A(BC)$ and $A(B+C) = AB + AC$.

The square matrix of dimension $n \times n$ having "1s" along the main diagonal and zeros elsewhere is called the *identity matrix* of dimension $n \times n$. In other words, a row-reduced square matrix is the identity matrix for that size square matrix. The identity matrix of dimension 3×3 is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The identity matrix has the property that $I_n A = A$ for any $n \times r$ matrix A , and $B I_n = B$ for any $s \times n$ matrix B . In particular, if A is a square matrix of dimension $n \times n$, then $I_n A = A I_n = A$.

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**Example 1
Scalar Multiplication**

Consider matrix $A = \begin{bmatrix} 1 & 1/3 \\ -6 & 8 \end{bmatrix}$. Find $6A$.

Multiply each element of A by six.

$$6A = \begin{bmatrix} 6 \cdot 1 & 6 \cdot (1/3) \\ 6 \cdot (-6) & 6 \cdot 8 \end{bmatrix}$$

$$6A = \begin{bmatrix} 6 & 2 \\ -36 & 48 \end{bmatrix}$$

**Example 2
Adding Matrices**

Consider the two matrices below.

$$A = \begin{bmatrix} 1 & 5 \\ -1 & 4 \end{bmatrix}, B = \begin{bmatrix} 9 & 3 \\ 0 & -2 \end{bmatrix}$$

What is the sum of A and B ? What is the difference of A and B ?

The sum of two matrices is the matrix whose elements are sums of the corresponding elements in the addends.

$$A + B = \begin{bmatrix} 1 & 5 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 9 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ -1 & 2 \end{bmatrix}$$

The difference of two matrices is the sum of the minuend and the product of -1 and the subtrahend.

$$\begin{aligned} A - B &= A + -1B = \begin{bmatrix} 1 & 5 \\ -1 & 4 \end{bmatrix} + -1 \begin{bmatrix} 9 & 3 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 5 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} -9 & -3 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -8 & 2 \\ -1 & 6 \end{bmatrix} \end{aligned}$$

Example 3
Multiplying Matrices

Consider the two matrices below.

$$G = \begin{bmatrix} 7 & 0 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 7 & 0 & 4 & 3 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

What is the product GH ?

If GH is defined, then H must have as many rows as G has columns. The product will have as many rows as G and as many columns as H . Since G is a 3×2 and H is a 2×4 , the product GH is defined as a 3×4 matrix.

$$GH = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$$

The element c_{11} is the sum of the products of the elements in row one of G and column one in H ; the element c_{12} is the sum of the products of the elements in row one of G and column two in H ; etc. Find the elements.

$$\begin{aligned} c_{11} &= 7(7) + 0(1) = 49 \\ c_{12} &= 7(0) + 0(-1) = 0 \\ c_{13} &= 7(4) + 0(2) = 28 \\ c_{14} &= 7(3) + 0(-2) = 21 \\ c_{21} &= 1(7) + (-1)(1) = 6 \\ c_{22} &= 1(0) + (-1)(-1) = 1 \\ c_{23} &= 1(4) + (-1)(2) = 2 \\ c_{24} &= 1(3) + (-1)(-2) = 5 \\ c_{31} &= 1(7) + 0(1) = 7 \\ c_{32} &= 1(0) + 0(-1) = 0 \\ c_{33} &= 1(4) + 0(2) = 4 \\ c_{34} &= 1(3) + 0(-2) = 3 \end{aligned}$$

Write the product.

$$GH = \begin{bmatrix} 49 & 0 & 28 & 21 \\ 6 & 1 & 2 & 5 \\ 7 & 0 & 4 & 3 \end{bmatrix}$$

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Perform the indicated operations.

$$\#1 \begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 2 & -2 \end{bmatrix}$$

$$\#2 \begin{bmatrix} -5 & 6 \\ -3 & .25 \\ 4 & .75 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 4 & 6 \\ -1 & .5 \end{bmatrix}$$

$$\#3 \begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 2 & -2 \end{bmatrix}$$

$$\#4 \begin{bmatrix} -1 & 6 \\ -2 & 3 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ -2 & 4 \\ 1 & 7 \end{bmatrix}$$

$$\#5 3 \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$\#6 -\frac{2}{3} \begin{bmatrix} 9 & 3 \\ -3 & 12 \\ 2 & 5 \end{bmatrix}$$

$$\#7 \begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 7 & 0 \\ 2 & -2 \end{bmatrix}$$

$$\#8 \begin{bmatrix} 5 & 3 \\ -1 & 2 \end{bmatrix} \cdot [2 \ 1]$$

$$\#9 \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\#10 [3 \ 5] \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\#11 \begin{bmatrix} -1 & 6 \\ -2 & 3 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & 0 & 1 \\ -2 & 4 & 1 & 0.5 \end{bmatrix}$$

$$\#12 \begin{bmatrix} -1 & 4 & \frac{2}{3} \\ -2 & 2 & 2 \\ 0 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#13 Show with one example that multiplication of matrices is not commutative.

#14 If the two matrices A and B below are equal, then solve for x , y , z , & w .

$$A = \begin{bmatrix} 4 & 12 \\ 3/4 & -7 \end{bmatrix}, B = \begin{bmatrix} \frac{5}{6}x + \frac{2}{3} & \frac{1}{2}y \\ z + \frac{1}{3} & 21w \end{bmatrix}$$

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$$\#1 \begin{bmatrix} 12 & 3 \\ 1 & 0 \end{bmatrix} \quad \#2 \begin{bmatrix} -7 & 4 \\ 1 & 6.25 \\ 3 & 1.25 \end{bmatrix} \quad \#3 \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix} \quad \#4 \begin{bmatrix} -2 & 3 \\ 0 & -1 \\ 1 & -5 \end{bmatrix} \quad \#5 \begin{bmatrix} 3 & 0 \\ 6 & -3 \end{bmatrix} \quad \#6 \begin{bmatrix} -6 & -2 \\ 2 & -8 \\ -4/3 & -10/3 \end{bmatrix}$$

$$\#7 \begin{bmatrix} 41 & -6 \\ -3 & -4 \end{bmatrix} \quad \#8 \text{ undefined} \quad \#9 \begin{bmatrix} 27 \\ -2 \end{bmatrix} \quad \#10 [27] \quad \#11 \begin{bmatrix} -13 & 21 & 6 & 2 \\ -8 & 6 & 3 & -.5 \\ -2 & 14 & 2 & 3 \end{bmatrix} \quad \#12 \begin{bmatrix} -1 & 4 & \frac{2}{3} \\ -2 & 2 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\#13 [3 \ 5] \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = [27], \begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot [3 \ 5] = \begin{bmatrix} 12 & 20 \\ 9 & 15 \end{bmatrix}, [27] \neq \begin{bmatrix} 12 & 20 \\ 9 & 15 \end{bmatrix} \therefore [3 \ 5] \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot [3 \ 5]$$

$$\#14 x = 4, y = 24, z = 5/12, w = -1/3$$

Study Exercise 4.5

Problems

For the problems below refer to the following matrices.

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

#1 Compute $3B$

#2 Compute $A + A$

#3 Compute $B \cdot A$.

#4 Compute $A \cdot B$.

Instruction: The Inverse of a Square Matrix

To define the *inverse of a matrix*, let A be a square matrix of dimension $n \times n$. The square matrix A^{-1} of dimension $n \times n$ such that $A^{-1}A = A A^{-1} = I_n$ is called the inverse of A . In other words if the product of square matrix A and another matrix of the same dimension, B , is the identity matrix, then A and B are inverses of one another. The following example shows that two matrices are inverses by finding their product.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(-2) + 2\left(\frac{3}{2}\right) & 1(1) + 2\left(-\frac{1}{2}\right) \\ 3(-2) + 4\left(\frac{3}{2}\right) & 3(1) + 4\left(-\frac{1}{2}\right) \end{bmatrix}$$

$$AB = \begin{bmatrix} (-2) + 3 & 1 + (-1) \\ (-6) + 6 & 3 + -2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By definition, B is the inverse of A , and can be written A^{-1} . Although matrix multiplication is not generally commutative, if $AB = I$, then $BA = I$ also. Hence, to verify that B is the inverse of A , it suffices to show that $AB = I$. Not every square matrix has an inverse. Matrices that do not have an inverse are called singular matrices.

To find the inverse of a nonsingular matrix, suppose that

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}.$$

If A^{-1} exists, then

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Multiplying the left side of the matrix equation yields:

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$$\begin{bmatrix} 1 \cdot a + 2 \cdot c & 1 \cdot b + 2 \cdot d \\ -1 \cdot a + 3 \cdot c & -1 \cdot b + 3 \cdot d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This matrix equation is equivalent to the two systems of linear equations

$$\begin{cases} a + 2c = 1 \\ -a + 3c = 0 \end{cases} \text{ and } \begin{cases} b + 2d = 0 \\ -b + 3d = 1. \end{cases}$$

These two systems of equations can be represented by the augmented matrices given by

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 3 & 0 \end{array} \right] \text{ and } \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 3 & 1 \end{array} \right]$$

Since the matrices of the coefficients of the two systems are identical, the two systems of simultaneous linear equations can be solved simultaneously by joining the coefficient matrix and the two columns of constants into a new augmented matrix that represents the two systems of equations.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array} \right]$$

The Gauss-Jordan method then solves the two systems simultaneously finding the elements of the inverse matrix.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array} \right] R_1 + R_2 \rightarrow R_2 \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array} \right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array} \right] \frac{1}{5} R_2 \rightarrow R_2 \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array} \right]$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array} \right] -2R_2 + R_1 \rightarrow R_1 \left[\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array} \right]$$

Thus, $a = 3/5$, $c = 1/5$, $b = -2/5$, and $d = 1/5$, yielding

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Multiplying A and A^{-1} yields I_2 and verifies that A^{-1} is the inverse of A .

This example suggests the general algorithm for computing the inverse of a square matrix of dimension $n \times n$:

Given the $n \times n$ matrix A :

1. Adjoin the $n \times n$ identity matrix I to obtain the augmented matrix $[A | I]$,
2. Use a sequence of row operations to reduce $[A | I]$ to the form $[I | A^{-1}]$.

This algorithm is convenient for finding the inverse of a square matrix that is larger than a 2×2 .

For 2×2 matrices there is an alternative definition for finding the inverse matrix:

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{If } ad - bc \text{ does not equal zero then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For example,

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$$A = \begin{bmatrix} 1 & 4 \\ 1 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1 \cdot 6 - 1 \cdot 4} \begin{bmatrix} 6 & -4 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 6 & -4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

NOTES

Example Exercises 4.6

Instruction: *Inverse Matrices*

Example 1
Finding the Inverse of a 2×2 Matrix

$$\text{Given } B = \begin{bmatrix} -5 & -\frac{1}{2} \\ h & 3 \end{bmatrix}, \text{ find } h \text{ such that } B^{-1} \text{ is defined.}$$

The inverse of a 2×2 matrix is defined below.

$$\text{If } A = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{c_{11} \cdot c_{22} - c_{21} \cdot c_{12}} \cdot \begin{bmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{bmatrix}.$$

Accordingly, the inverse of a 2×2 matrix is defined if the determinant of the matrix does not equal zero.

$$-5(3) - h\left(-\frac{1}{2}\right) \neq 0$$

$$-15 + \frac{h}{2} \neq 0$$

$$\frac{h}{2} \neq 15$$

$$h \neq 30$$

The inverse of B is defined for all values of h except 30.

Example Exercises 4.6

Example 2
Finding the Inverse of a 3×3 Matrix

$$\text{Given } C = \begin{bmatrix} 6 & 2 & -1 \\ 2 & 1 & 0 \\ 2 & 3 & 9 \end{bmatrix}, \text{ find } C^{-1}.$$

Write the augmented matrix $[C \ I_{3 \times 3}]$.

$$\begin{bmatrix} 6 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 9 & 0 & 0 & 1 \end{bmatrix}$$

Use row operations to reduce $[C \ I_{3 \times 3}]$ to $[I_{3 \times 3} \ C^{-1}]$.

$$\begin{aligned} & \begin{bmatrix} 6 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 9 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & -1/6 & 1/6 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 9 & 0 & 0 & 1 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 1/3 & -1/6 & 1/6 & 0 & 0 \\ 0 & 1/3 & 1/3 & -1/3 & 1 & 0 \\ 0 & 7/3 & 26/3 & -1/3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & -1/6 & 1/6 & 0 & 0 \\ 0 & 1 & 1 & -1 & 3 & 0 \\ 0 & 7/3 & 26/3 & -1/3 & 0 & 1 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 0 & -1/2 & 1/2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 3 & 0 \\ 0 & 0 & 19/3 & 2 & -7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 & 1/2 & -1 & 0 \\ 0 & 1 & 1 & -1 & 3 & 0 \\ 0 & 0 & 1 & 6/19 & -21/19 & 3/19 \end{bmatrix} \\ & \sim \begin{bmatrix} 1 & 0 & 0 & 25/38 & -59/38 & 3/38 \\ 0 & 1 & 0 & -25/19 & 78/19 & -3/19 \\ 0 & 0 & 1 & 6/19 & -21/19 & 3/19 \end{bmatrix}, C^{-1} = \begin{bmatrix} 25/38 & -59/38 & 3/38 \\ -25/19 & 78/19 & -3/19 \\ 6/19 & -21/19 & 3/19 \end{bmatrix} \end{aligned}$$

Example Exercises 4.6

Example 3 Verifying Inverses

Given $L = \begin{bmatrix} -2 & 4 \\ h & 3 \end{bmatrix}$ & $K = \begin{bmatrix} -3/2 & 2 \\ -1/2 & 1 \end{bmatrix}$, find h such that L and K are inverses.

Recall that if $L_{2 \times 2}$ and $K_{2 \times 2}$ are inverses of each other, then $L_{2 \times 2} K_{2 \times 2} = I_{2 \times 2}$. Assume the elements of $L_{2 \times 2} K_{2 \times 2}$ are c_{11} , c_{12} , c_{21} , and c_{22} , then

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Show that $c_{11} = 1$ and $c_{12} = 0$.

c_{11} = sum of products of elements in row one of L and corresponding elements in column one of K

$$c_{11} = -2(-3/2) + 4(-1/2) = 3 - 2 = 1$$

c_{12} = sum of products of elements in row one of L and corresponding elements in column two of K

$$c_{12} = -2(2) + 4(1) = -4 + 4 = 0$$

Find c_{21} and c_{22} in terms of h .

c_{21} = sum of products of elements in row two of L and corresponding elements in column one of K

$$c_{21} = h(-3/2) + 3(-1/2) = -\frac{3}{2}h - \frac{3}{2}$$

c_{22} = sum of products of elements in row two of L and corresponding elements in column two of K

$$c_{22} = h(2) + 3(1) = 2h + 3$$

Set c_{21} equal to zero and c_{22} equal to one. Solve the two equations. If h is the same solution for each equation, then the solution is the value of h such that L is the inverse of K .

$$\begin{aligned} -\frac{3}{2}h - \frac{3}{2} &= 0 & 2h + 3 &= 1 \\ -\frac{3}{2}h &= \frac{3}{2} & 2h &= -2 \\ h &= -1 & h &= -1 \end{aligned}$$

L and K are inverses when $h = -1$.

Practice Set 4.6

#1 If $A = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$, find A^{-1} .

#2 If $B = \begin{bmatrix} 9 & 2/3 \\ 7 & -2 \end{bmatrix}$, find B^{-1} .

#3 If $G = \begin{bmatrix} 2 & -7 \\ 1 & -3 \end{bmatrix}$, find G^{-1} .

#4 If $D = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, find D^{-1} .

#5 If $R = \begin{bmatrix} 2 & -1 & 3 \\ 1 & -3 & 2 \\ 1 & 0 & 2 \end{bmatrix}$, find R^{-1} .

#6 If $Y = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$, find Y^{-1} .

#7 Use the product $C \cdot D$ to verify that C and D are inverses of each other.

$$C = \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix}, D = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$$

#8 Use the product $A \cdot B$ to verify that A and B are inverses of each other.

$$A = \begin{bmatrix} -1 & 4 & 5 \\ -2 & 2 & 2 \\ 0 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} 4/11 & -15/22 & -1/11 \\ 5/11 & -5/22 & -4/11 \\ -1/11 & 1/22 & 3/11 \end{bmatrix}$$

ANSWERS

$$\#1 A^{-1} = \begin{bmatrix} 3/7 & -2/7 \\ -1/7 & 3/7 \end{bmatrix} \quad \#2 B^{-1} = \begin{bmatrix} 3/34 & -1/34 \\ 21/68 & -27/68 \end{bmatrix} \quad \#3 G^{-1} = \begin{bmatrix} -3 & 7 \\ -1 & 2 \end{bmatrix} \quad \#4 D^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$$

$$\#5 R^{-1} = \begin{bmatrix} 2 & -2/3 & -7/3 \\ 0 & -1/3 & 1/3 \\ -1 & 1/3 & 5/3 \end{bmatrix} \quad \#6 Y^{-1} = \begin{bmatrix} 0 & 0 & 1/3 \\ -1/2 & -1/2 & 1/2 \\ 1 & 0 & -1/3 \end{bmatrix} \quad \#7 \begin{bmatrix} -1 & 3 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\#8 \begin{bmatrix} -1 & 4 & 5 \\ -2 & 2 & 2 \\ 0 & 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4/11 & -15/22 & -1/11 \\ 5/11 & -5/22 & -4/11 \\ -1/11 & 1/22 & 3/11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Study Exercise 4.6

Problems

#1 If $A = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$, find A^{-1} .

#2 If $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$, find B^{-1} .