

College Algebra

Instruction: *Determinants*

As shown in Section 4.2, matrices arise in connection with systems of linear equations like the system below.

$$\begin{aligned}a_1x_1 + b_1x_2 &= c_1 \\ a_2x_1 + b_2x_2 &= c_2\end{aligned}$$

The augmented matrix for the system above is below.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

The coefficient matrix does not include the column of constants as shown below.

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

If a matrix is a square matrix (the number of rows is equal to the number of columns) as above, then there is often a number associated with the matrix called its determinant.

A determinant is a real number associated with a square matrix and is indicated by enclosing the array between two vertical bars. For a matrix A , the corresponding determinant is designated as $\det(A)$ and is read "determinant of A ." For example,

$$\text{matrix } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

A determinant may be of any size $n \times n$ where n is a positive integer greater than one. This discussion will be restricted to 2×2 and 3×3 determinants.

Every determinant with real elements has a real value. The method for finding the value of 3×3 determinants involves finding the value of 2×2 determinants. Determinants of larger matrices can be evaluated by using techniques similar to those shown here.

The value of a 2×2 determinant is the difference between the products of the diagonals as given by the following definition:

$$\text{For the square matrix } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

One method of evaluating 3×3 determinants is called *expanding by minors*. In this method, one row is chosen and each element in that row has a *minor*. Each minor is found by mentally crossing out both the row and column (shown below with dotted lines) that contain that element. The minors of the elements of the first row are shown here:

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$$\begin{array}{ccc}
 \begin{array}{|c|c|c|} \hline c_{11} & c_{12} & c_{13} \\ \hline c_{21} & c_{22} & c_{23} \\ \hline c_{31} & c_{32} & c_{33} \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline c_{22} & c_{23} \\ \hline c_{32} & c_{33} \\ \hline \end{array} \leftarrow \text{minor of } c_{11} \\
 \\
 \begin{array}{|c|c|c|} \hline c_{11} & c_{12} & c_{13} \\ \hline c_{21} & c_{22} & c_{23} \\ \hline c_{31} & c_{32} & c_{33} \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline c_{21} & c_{23} \\ \hline c_{31} & c_{33} \\ \hline \end{array} \leftarrow \text{minor of } c_{12} \\
 \\
 \begin{array}{|c|c|c|} \hline c_{11} & c_{12} & c_{13} \\ \hline c_{21} & c_{22} & c_{23} \\ \hline c_{31} & c_{32} & c_{33} \\ \hline \end{array} & \rightarrow & \begin{array}{|c|c|} \hline c_{21} & c_{22} \\ \hline c_{31} & c_{32} \\ \hline \end{array} \leftarrow \text{minor of } c_{13}
 \end{array}$$

Finding the product of each element and its minor and then adding these products with an adjustment of alternating signs finds the value of a determinant as given by the following definition:

$$\begin{aligned}
 A &= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}, \det(A) = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \\
 &= c_{11} \cdot \begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix} - c_{12} \cdot \begin{vmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{vmatrix} + c_{13} \cdot \begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix}
 \end{aligned}$$

The value of the determinant may also be found by expanding by the minors of any other row or any column. Each minor is the product of its corresponding element and -1 to the $r + c$ power, $(-1)^{r+c}$, where r is the row and c is the column of the element. The factor $(-1)^{r+c}$ always produces the following pattern of signs:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

For instance, the value of a 3×3 determinant could be found by expanding by the minors of row two according to the chart above: $\det(A) = -c_{21}(\text{minor of } c_{21}) + c_{22}(\text{minor of } c_{22}) - c_{23}(\text{minor of } c_{23})$.

Consider matrix C .

$$C = \begin{bmatrix} 5 & 1 & -4 \\ 2 & 6 & 3 \\ 2 & 2 & 1 \end{bmatrix}$$

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The determinant of C is found below by expanding by the minors of row one.

$$\begin{aligned}\det(C) &= 5 \begin{vmatrix} 6 & 3 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + (-4) \begin{vmatrix} 2 & 6 \\ 2 & 2 \end{vmatrix} \\ \det(C) &= 5(6 \cdot 1 - 2 \cdot 3) - 1(2 \cdot 1 - 2 \cdot 3) - 4(2 \cdot 2 - 2 \cdot 6) \\ \det(C) &= 5(6 - 6) - 1(2 - 6) - 4(4 - 12) \\ \det(C) &= 5(0) - 1(-4) - 4(-8) \\ \det(C) &= 36\end{aligned}$$

In the next example, the process for finding a 3×3 determinant is simplified by transforming the matrix with elementary row and column operations first.

$$D = \begin{bmatrix} 6 & -2 & 4 \\ -2 & 9 & -1 \\ -3 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 4 \\ -2 & 9 & -1 \\ -3 & 2 & -1 \end{bmatrix} \xrightarrow{3 \cdot R_2 + R_1 \rightarrow R_1} \begin{bmatrix} 0 & 25 & 1 \\ -2 & 9 & -1 \\ -3 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 25 & 1 \\ -2 & 9 & -1 \\ -3 & 2 & -1 \end{bmatrix} \xrightarrow{-25C_3 + C_2 \rightarrow C_2} \begin{bmatrix} 0 & 0 & 1 \\ -2 & 34 & -1 \\ -3 & 27 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 4 \\ -2 & 9 & -1 \\ -3 & 2 & -1 \end{bmatrix} \xleftrightarrow{\text{row-column equivalent}} \begin{bmatrix} 0 & 0 & 1 \\ -2 & 34 & -1 \\ -3 & 27 & -1 \end{bmatrix}$$

$$\det(D) = \begin{vmatrix} 6 & -2 & 4 \\ -2 & 9 & -1 \\ -3 & 2 & -1 \end{vmatrix}$$

$$\det(D) = \begin{vmatrix} 0 & 0 & 1 \\ -2 & 34 & -1 \\ -3 & 27 & -1 \end{vmatrix}$$

$$\det(D) = 0 \begin{vmatrix} 34 & -1 \\ 27 & -1 \end{vmatrix} - 0 \begin{vmatrix} -2 & -1 \\ -3 & -1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 34 \\ -3 & 27 \end{vmatrix}$$

$$\det(D) = 1(-2 \cdot 27 - (-3) \cdot 34)$$

$$\det(D) = -54 + 102$$

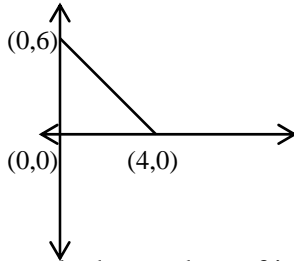
$$\det(D) = 48$$

Instruction: Some Applications of Determinants

It can be shown that the area of a triangle formed by the points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) is

given by the absolute value of $\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

For example, consider the triangle determined by $(0,0)$, $(0,6)$, $(4,0)$ as shown here.



Recall that the area of a triangle equals the product of its base, height, and one-half. Accordingly, the area of the triangle is $\frac{1}{2} \times 4 \times 6 = 12$. Substituting the coordinates of the four vertices of the triangle into the determinant formula yields the same area:

$$\text{Area} = \frac{1}{2} \cdot \text{ABS} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 6 & 1 \\ 4 & 0 & 1 \end{vmatrix} = \frac{1}{2} \cdot \text{ABS}[4(0 - 6)] = \frac{1}{2} \cdot \text{ABS}(-24) = \frac{1}{2} \cdot 24 = 12.$$

(Note the use of ABS for absolute value. The determinant formula presents some notational difficulties since the symbol for absolute value, two vertical segments, is appropriated to indicate the determinant.)

The determinant formula can also be used to show that three points are collinear.

Suppose one calculates the absolute value of $\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ and gets zero. This means the area

of the triangle is zero. If the area of a triangle is zero, then the three points must lie in a line. Thus, the determinant formula for the area of a triangle provides a check for collinearity. Three

points, (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , are collinear if and only if the absolute value of $\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

Determinants present an alternate method for finding the equation of a line through two points. In a Cartesian plane, lines have equations of the form $Ax + By + C = 0$, and any equation of the form $Ax + By + C = 0$ is a line. Consider the equation attained from the determinant on the next page.

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$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$x \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix} - y \begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix} + 1 \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0$$

$$x(y_1 - y_2) - y(x_1 - x_2) + (x_1 y_2 - x_2 y_1) = 0$$

For fixed x_1, x_2, y_1, y_2 this equation is an equation of the form $Ax + By + C = 0$, i.e., a line. Furthermore, this line contains the points (x_1, y_1) and (x_2, y_2) . To check, let $x = x_1$ and $y = y_1$:

$$x_1(y_1 - y_2) - y_1(x_1 - x_2) + (x_1 y_2 - x_2 y_1) = 0$$

$$x_1 y_1 - x_1 y_2 - x_1 y_1 + x_2 y_1 + x_1 y_2 - x_2 y_1 = 0$$

$$x_1 y_1 - x_1 y_1 - x_1 y_2 + x_1 y_2 + x_2 y_1 - x_2 y_1 = 0$$

$$0 = 0$$

Thus, determinants present an alternate method for finding the equation of a line through two given points. For example, to find the equation of the line that passes through $(1, 5)$ and $(3, 2)$,

substitute into the equation $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ as shown below.

$$\begin{vmatrix} x & y & 1 \\ 1 & 5 & 1 \\ 3 & 2 & 1 \end{vmatrix} = 0$$

$$x \begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = 0$$

$$x(5 - 2) - y(1 - 3) + (2 - 15) = 0$$

$$x(3) - y(-2) + (-13) = 0$$

$$3x + 2y - 13 = 0$$

It is not important to expand using the first row. Any row or column will work.

A similar determinant technique finds the equation of a parabola passing through three given points. In general, assume three points on a parabola are (x_1, y_1) , (x_2, y_2) , (x_3, y_3) . Then,

$$\begin{vmatrix} x^2 & x & y & 1 \\ x_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

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gives the equation of the parabola passing through (x_1, y_1) , (x_2, y_2) , (x_3, y_3) . For example, consider a parabola passing through the points $(3, 2)$, $(4, 8)$, and $(1, 5)$. Substitute these coordinate values into the formula above to obtain:

$$\begin{vmatrix} x^2 & x & y & 1 \\ 3^2 & 3 & 2 & 1 \\ 4^2 & 4 & 8 & 1 \\ 1^2 & 1 & 5 & 1 \end{vmatrix} = 0$$

Simplify, then expand by minors using the first row as shown below.

$$\begin{vmatrix} x^2 & x & y & 1 \\ 9 & 3 & 2 & 1 \\ 16 & 4 & 8 & 1 \\ 1 & 1 & 5 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & 2 & 1 \\ 4 & 8 & 1 \\ 1 & 5 & 1 \end{vmatrix} x^2 - \begin{vmatrix} 9 & 2 & 1 \\ 16 & 8 & 1 \\ 1 & 5 & 1 \end{vmatrix} x + \begin{vmatrix} 9 & 3 & 1 \\ 16 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} y - \begin{vmatrix} 9 & 3 & 2 \\ 16 & 4 & 8 \\ 1 & 1 & 5 \end{vmatrix} = 0$$

Evaluating each of the 3×3 determinants obtains: $15x^2 - 69x - 6y + 84 = 0$, which can be written as $y = \frac{5}{2}x^2 - \frac{23}{2}x + 14$.

To check, substitute each of the three given points and note that an identity, in each case, is obtained.

Example Exercises 4.3

Instruction: *Determinants*

Example 1 Finding the Determinant of a 2×2 Matrix

$$\text{Given } B = \begin{bmatrix} -5 & -\frac{1}{2} \\ 6 & 3 \end{bmatrix}, \text{ find } \det(B).$$

$$\det(B) = \begin{vmatrix} -5 & -\frac{1}{2} \\ 6 & 3 \end{vmatrix}$$

$$\det(B) = -5(3) - 6\left(-\frac{1}{2}\right)$$

$$\det(B) = -15 + 3$$

$$\det(B) = -12.$$

Example 2 Finding the Determinant of a 3×3 Matrix

$$\text{Given } C = \begin{bmatrix} 6 & 2 & -1 \\ 2 & 1 & 0 \\ 2 & 3 & 9 \end{bmatrix}, \text{ find } \det(C).$$

Since row two has a zero and two positive elements, expanding by the minors of row two is convenient. Recall the sign pattern for row two $(-, +, -)$.

$$\det(C) = -2 \begin{vmatrix} 2 & -1 \\ 3 & 9 \end{vmatrix} + 1 \begin{vmatrix} 6 & -1 \\ 2 & 9 \end{vmatrix} - 0 \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix}$$

Calculate the determinant of each minor and simplify.

$$\det(C) = -2(2 \cdot 9 - 3 \cdot -1) + 1(6 \cdot 9 - 2 \cdot -1) - 0(6 \cdot 3 - 2 \cdot 2)$$

$$\det(C) = -2(18 + 3) + 1(54 + 2) - 0(18 - 4)$$

$$\det(C) = -2(21) + 1(56) - 0(14)$$

$$\det(C) = -42 + 56 - 0$$

$$\det(C) = 14$$

Practice Set 4.3A

#1 If $A = \begin{bmatrix} 3 & -1 \\ 9 & 2 \end{bmatrix}$, find $\det(A)$.

#2 Evaluate $\begin{vmatrix} 2 & 4 & 6 \\ -1 & 3 & 3 \\ 5 & -2 & 2 \end{vmatrix}$.

#3 Evaluate $\begin{vmatrix} 5 & 8 \\ 6 & 10 \end{vmatrix}$.

#4 Show with any example that the determinant of a matrix is zero if every element in a row or column of that matrix is zero.

#5 Show with any example that if every element of a row or column of a matrix is multiplied by a real number k , then the determinant of the new matrix is k times the determinant of the original matrix.

ANSWERS

#1 $\det(A) = 15$

#2 14

#3 2

#4 Consider $\begin{vmatrix} 0 & 0 \\ 7 & 1 \end{vmatrix} = 0 \cdot 1 - 7 \cdot 0 = 0$

#5 Compare $\begin{vmatrix} 4 & 2 \\ -1 & 1 \end{vmatrix} = 4 \cdot 1 - (-1 \cdot 2) = 6$ to $\begin{vmatrix} 8 & 4 \\ -1 & 1 \end{vmatrix} = 8 \cdot 1 - (-1 \cdot 4) = 12$

Practice Set 4.3B

- #1 Use determinants to find the area of a triangle formed by $(-3, -2)$, $(2,2)$, $(4.5,-2)$.
- #2 Use determinants to find the area of a triangle formed by $(2,3)$, $(-4,2)$, $(5,3)$.
- #3 Use determinants to find the area of a triangle formed by $(-4, -3)$, $(0,5)$, $(7,1)$.
- #4 Use determinants to find the equation of a line passing through $(3,2)$ and $(-4, -5)$.
- #5 Use determinants to find the equation of a line passing through $(1,5)$ and $(-1, -3)$.
- #6 Use determinants to find the equation of a line passing through $(0,6)$ and $(-3, -8)$.
- #7 Use determinants to find the equation of a parabola passing through $(0,-7)$, $(1,0)$, $(2,11)$.
- #8 Use determinants to find the equation of a parabola passing through $(0,4)$, $(1,1)$, $(-1,1)$.

ANSWERS

#1 Area = 15 square units

#2 Area = 1.5 square units

#3 Area = 36 square units

#4 $7x - 7y - 7 = 0$, or, more simply after dividing each term of the equation by seven, $x - y - 1 = 0$

#5 $4x - y + 1 = 0$

#6 $14x - 3y + 18 = 0$

#7 $y = 2x^2 + 5x - 7$

#8 $y = 3x^2 - 4$

Study Exercise 4.3

Problems

#1 Evaluate $\begin{vmatrix} -2 & 2 \\ 2 & -12 \end{vmatrix}$.

#2 If $\begin{vmatrix} 3 & 1 \\ 1 & x \end{vmatrix} = 17$, find the value of x .

#3 If $\begin{vmatrix} x & -1 \\ 1 & x \end{vmatrix} = 17$, find the value(s) of x .

#4 Use determinants to find the equation of a line passing through (1,1) and (2,3).