Lecture 4.1

College Algebra

Instruction: ***Solving Systems of Linear Equations (Three Variables)***

This lecture discusses solving systems of linear equations with three variables. If the system has two equations each representing a distinct plane, two results can be obtained:

1. no solution
2. infinite solutions that coincide with a line.

If the system has three equations each representing a distinct plane, three results can be obtained:

1. one solution
2. no solution
3. infinite solutions that coincide with a line.

Following the rubric below will help solve a system of equations with three variables.

1. Select two equations and eliminate one variable using the addition method.
2. Select two other equations and eliminate the same variable.
3. Steps 1 and 2 give two linear equations in two variables. Solve these equations either by addition or substitution.
4. Back substitute the values found in Step 3 into any one of the original equations to find the value of the third value.
5. Check the solution with the original system of equations.

In cases where three or more equations have only two variables, using a substitution method--in a sense, going directly to step four--is more viable than the elimination method outlined above.

Example 1

$$\begin{cases} 2x+3y-z=16 & \text{Equation } A \\ x-y+3z=-9 & \text{Equation } B \\ 5x+2y-z=15 & \text{Equation } C \end{cases}$$

Step 1 Multiply *Equation B* by –2 then add it to *Equation A* so that the *x*-variable will be eliminated.

$$\begin{cases} 2x+3y-z=16 \\ -2(x-y+3z=-9) \end{cases} \qquad \begin{array}{r} 2x+3y-z=16 \\ \underline{-2x+2y-6z=18} \\ 5y-7z=34 \end{array}$$

Step 2 Use two different equations, this time *Equation B* and *Equation C*, to eliminate the same variable.

$$\begin{cases} -5(x-y+3z=-9) \\ 5x+2y-z=15 \end{cases} \qquad \begin{array}{r} -5x+5y-15z=45 \\ \underline{5x+2y-z=15} \\ 7y-16z=60 \end{array}$$

Step 3 Group the two linear equations in two variables from steps one and two as a separate system of equations and solve for the two variables, y and z.

$$\begin{cases} 5y - 7z = 34 \\ 7y - 16z = 60 \end{cases}$$

$$\begin{cases} -7(5y - 7z = 34) \\ 5(7y - 16z = 60) \end{cases} \qquad \begin{aligned} -35y + 49z &= -238 \\ \underline{35y - 80z} &\underline{= 300} \\ -31z &= 62 \\ z &= -2 \end{aligned}$$

Back substitute $z = -2$ in order to solve for y.

$$5y - 7(-2) = 34$$
$$5y + 14 = 34$$
$$5y = 34 - 14$$
$$5y = 20$$
$$y = 4$$

Step 4 Back substitute the values found above, $y = 4$ and $z = -2$, into any one of the original equations to find the value of the third variable.

$$2x + 3y - z = 16$$
$$2x + 3(4) - (-2) = 16$$
$$2x + 12 + 2 = 16$$
$$2x + 14 = 16$$
$$2x = 16 - 14$$
$$2x = 2$$
$$x = 1$$

The solution is (1,4,-2).

Step 5 Check the solution in the three equations.

$$\begin{cases} 2x+3y-z=16 \\ x-y+3z=-9 \\ 5x+2y-z=15 \end{cases}$$

$$\begin{cases} 2(1)+3(4)-(-2)=16 \\ (1)-(4)+3(-2)=-9 \\ 5(1)+2(4)-(-2)=15 \end{cases}$$

$$\begin{cases} 2+12+2=16 \\ 1-4-6=-9 \\ 5+8+2=15 \end{cases}$$

Example 2

$$\begin{cases} x-y+2z=-4 & \text{Equation } A \\ 2x+3y+z=\frac{1}{2} & \text{Equation } B \\ x+4y-2z=4 & \text{Equation } C \end{cases}$$

Step 1 Multiply *Equation A* by –1 then add it to *Equation C* so that the *x*-variable will be eliminated.

$$\begin{cases} -1(x-y+2z=-4) \\ \quad x+4y-2z=4 \end{cases} \qquad \begin{array}{r} -x+y-2z=4 \\ \underline{x+4y-2z=4} \\ 5y-4z=8 \end{array}$$

Step 2 Use two different equations, this time *Equation A* and *Equation B,* to eliminate the same variable.

$$\begin{cases} -2(x-y+2z=-4) \\ \quad 2x+3y+z=\frac{1}{2} \end{cases} \qquad \begin{array}{r} -2x+2y-4z=8 \\ \underline{2x+3y+z=\frac{1}{2}} \\ 5y-3z=8\frac{1}{2} \end{array}$$

Step 3 Group the two linear equations in two variables from steps one and two as a separate system of equations and solve for the two variables, *y* and *z*.

$$\begin{cases} 5y - 4z = 8 \\ 5y - 3z = 8\frac{1}{2} \end{cases}$$

$$\begin{cases} -1(5y - 4z = 8) \\ \quad 5y - 3z = 8\frac{1}{2} \end{cases}$$

$$\begin{array}{r} -5y + 4z = -8 \\ 5y - 3z = 8\frac{1}{2} \\ \hline z = \frac{1}{2} \end{array}$$

Back substitute $z = {}^1/_2$
in order to solve for y.

$$5y - 4\left(\frac{1}{2}\right) = 8$$
$$5y - 2 = 8$$
$$5y = 8 + 2$$
$$5y = 10$$
$$y = 2$$

Step 4 Back substitute the values found above, $y = 2$ and $z = {}^1/_2$, into any one of the original equations to find the value of the third variable.

$$x - y + 2z = -4$$
$$x - (2) + 2\left(\frac{1}{2}\right) = -4$$
$$x - 2 + 1 = -4$$
$$x - 1 = -4$$
$$x = -4 + 1$$
$$x = -3$$

The solution is $(-3, 2, {}^1/_2)$.

Step 5 Check the solution in the three equations.

$$\begin{cases} x - y + 2z = -4 \\ 2x + 3y + z = \dfrac{1}{2} \\ x + 4y - 2z = 4 \end{cases}$$

$$\begin{cases} (-3) - (2) + 2\left(\dfrac{1}{2}\right) = -4 \\ 2(-3) + 3(2) + \left(\dfrac{1}{2}\right) = \dfrac{1}{2} \\ (-3) + 4(2) - 2\left(\dfrac{1}{2}\right) = 4 \end{cases}$$

$$\begin{cases} -3 - 2 + 1 = -4 \\ -6 + 6 + \dfrac{1}{2} = \dfrac{1}{2} \\ -3 + 8 - 1 = 4 \end{cases}$$

Example 3

$$\begin{cases} 3x - 5y + z = 6 & \text{Equation } A \\ x - y + 3z = -1 & \text{Equation } B \\ 2x - 2y + 6z = 5 & \text{Equation } C \end{cases}$$

Step 1 Multiply *Equation A* by –3 then add it to *Equation B* in order to eliminate the *z*-variable.

$$\begin{cases} -3(3x - 5y + z = 6) \\ \quad x - y + 3z = -1 \end{cases} \qquad \begin{array}{r} -9x + 15y - 3z = -18 \\ \underline{x - \ \ y + \ \ 3z = -1} \\ -8x + 14y \qquad = -19 \end{array}$$

Step 2 Use different equations, *Equation B* and *Equation C*, in order to eliminate the same variable.

$$\begin{cases} -2(x - y + 3z = -1) \\ \quad 2x - 2y + 6z = 5 \end{cases} \qquad \begin{array}{r} -2x + 2y - 6z = 2 \\ \underline{2x - 2y + 6z = 5} \\ 0 = 7 \end{array}$$

The resulting equation without variables (0 = 7) is false indicating that the system does not have a solution. All three planes do not intersect at any given point.

Example 4

$$\begin{cases} x - 2y + z = 4 & \text{Equation } A \\ x - y - 4z = 1 & \text{Equation } B \\ 2x - 4y + 2z = 8 & \text{Equation } C \end{cases}$$

Step 1 Add *Equation A* to *Equation B* in order to eliminate the *x*-variable.

$$\begin{cases} -1(x-2y+z=4) \\ \quad x-y-4z=1 \end{cases} \qquad \begin{array}{r} -x+2y-z=-4 \\ \underline{x-y-4z=1} \\ y-5z=-3 \end{array}$$

Step 2 Use *Equation C* and *Equation A* to eliminate the same variable.

$$\begin{cases} -2(x-2y+z=4) \\ \quad 2x-4y+2z=8 \end{cases} \qquad \begin{array}{r} -2x+4y-2z=-8 \\ \underline{2x-4y+2z=8} \\ 0=0 \end{array}$$

The resulting equation without variables (0=0) is true, indicating that the system may have an infinite number of solutions. A general solution for the infinite solutions can be found. To find the general solution, first let $z = t$. Second, substitute t for z in the two variable equation found in step one, and solve for y:

$$y-5z=-3$$
$$y-5t=-3$$
$$y=-3+5t$$

Third, substitute t for z and $-3 + 5t$ for y in any one of the original equations:

$$x-2y+z=4$$
$$x-2(-3+5t)+t=4$$
$$x+6-10t+t=4$$
$$x+6-9t=4$$
$$x=4-6+9t$$
$$x=-2+9t$$

The general solution for the system in terms of t is $(-2+9t, -3+5t, t)$. To find particular solutions to the system of equations, simply substitute any real number for t. For example, let $t = 0$, then a particular solution would be $(-2,-3,0)$. Or, let $t = 1$, then a particular solution would be:

$$(-2+9t,\ -3+5t,\ t)$$
$$(-2+9\cdot 1,\ -3+5\cdot 1,\ 1)$$
$$(-2+9,\ -3+5,\ 1)$$
$$(7,\ 2,\ 1).$$

These particular solutions can be checked by substituting them into each of the equations in the original system of equations.

Example 5

$$x+3z=25 \quad \text{Equation } A$$
$$2x+y=11 \quad \text{Equation } B$$
$$y-z=1 \quad \text{Equation } C$$

Since the system contains three equations each with only two of three variables, employ substitution instead of elimination. Solve two of the equations in terms of a variable present in the third equation.

$$x+3z=25 \qquad y-z=1$$
$$x=25-3z \qquad y=1+z$$

Substitute into the third equation.

$$2x+y=11$$
$$2(25-3z)+(1+z)=11$$
$$50-6z+1+z=11$$
$$51-5z=11$$
$$-5z=-40$$
$$z=8$$

Back substitute.

$$x=25-3(8)=1$$
$$y=1+(8)=9$$

Write the solution in ordered triple form: (1,9,8).

Instruction: ***Solving Systems of Linear Equations with Three Variables***

Example 1
Solving a System of Linear Equations Using Elimination and Back Substitution

Solve the system of equations below.

$$\begin{aligned} 3x + y + 4z &= -5 \\ 2x - 3y + z &= 0 \\ x + 5y - z &= 3 \end{aligned}$$

Eliminate one variable from first two equations by adding the equations to multiples of the third equation.

$$-3(x + 5y - z = 3) \rightarrow -3x - 15y + 3z = -9$$
$$\underline{3x + y + 4z \quad = -5}$$
$$-14y + 7z = -14$$

$$-2(x + 5y - z = 3) \rightarrow -2x - 10y + 2z = -6$$
$$\underline{2x - 3y + z \quad = 0}$$
$$-13y + 3z = -6$$

Rewrite the system of equations using the two new equations with two variables.

$$\begin{aligned} x + 5y - z &= 3 \\ -14y + 7z &= -14 \\ -13y + 3z &= -6 \end{aligned}$$

Eliminate a second variable from the new equation two by adding a multiple of the new equation two to a multiple of the new equation one.

$$13(-14y + 7z = -14) \rightarrow -182y + 91z = -182$$
$$-14(-13y + 3z = -6) \rightarrow \underline{182y - 42z = 84}$$
$$49z = -98$$
$$z = -2$$

Rewrite the system of equations using the new equation with one variable.

Example Exercises 4.1

$$\begin{aligned} x + 5y - z &= 3 \\ -14y + 7z &= -14 \\ z &= -2 \end{aligned}$$

Back substitute the z-value into the equation with only two variables and solve.

$$\begin{aligned} &-14y + 7(-2) = -14 \\ &-14y - 14 = -14 \\ &-14y = 0 \\ &y = 0 \end{aligned}$$

Back substitute the z and y-values into the equation with three variables and solve.

$$\begin{aligned} &x + 5(0) - (-2) = 3 \\ &x + 2 = 3 \\ &x = 1 \end{aligned}$$

State the solution in ordered triple form: $(1, 0, -2)$.

Example 2
Finding a General Solution for Systems with Infinite Many Solutions

Solve the system of equations below.

$$3x + 6y - 9z = 15$$
$$2x + 4y - 6z = 10$$
$$-2x - 3y + 4z = -6$$

Eliminate one variable from the last two equations by adding the equations to multiples of the first equation.

$$\frac{2}{3}(3x + 6y - 9z = 15) \rightarrow 2x + 4y - 6z = 10$$
$$\underline{-2x - 3y + 4z = -6}$$
$$y - 2z = 4$$

$$-\frac{2}{3}(3x + 6y - 9z = 15) \rightarrow -2x - 4y + 6z = -10$$
$$\underline{2x + 4y - 6z = 10}$$
$$0 = 0$$

When the operations with equations eliminates all variables, the system has either no solution or infinite many solutions. If the resulting equation is true as is 0 = 0, then the system has infinite many solutions. The infinite many solutions can be represented using a parameter. Let $z = t$, the parameter. Substitute the parameter for z into the equation with two variables and solve for y in terms of the parameter.

$$y - 2z = 4$$
$$y - 2t = 4$$
$$y = 4 + 2t$$

Substitute the parameter for z and the parametric expression for y into the equation with three variables and solve for x in terms of the parameter.

$$3x + 6y - 9z = 15$$
$$3x + 6(4 + 2t) - 9t = 15$$
$$3x + 24 + 12t - 9t = 15$$
$$3x + 3t = 15 - 24$$
$$3x = -9 - 3t$$
$$x = -3 - t$$

Write the solution in terms of the parameter in ordered triple form: $(-3-t,\ 4+2t,\ t)$. This represents a general solution. Any one of the infinite many particular solutions can be attained by assigning a value to the parameter. For instance, if $t = 0$, then the solution is $(-3,\ 4,\ 0)$.

Example 3
Solving a System of Linear Equations Using Substitution

Solve the system of equations below.

$$\begin{aligned} x+2z&=-4\\ 2x-y&=3\\ 3y-z&=6 \end{aligned}$$

Solve two of the equations in terms of a variable present in the third equation.

$$\begin{aligned} x+2z&=-4\\ x&=-4-2z \end{aligned}$$

$$\begin{aligned} 3y-z&=6\\ 3y&=6+z\\ y&=\frac{1}{3}(6+z)\\ y&=2+\frac{1}{3}z \end{aligned}$$

Substitute into the third equation.

$$\begin{aligned} &2x-y=3\\ &2(-4-2z)-\left(2+\frac{1}{3}z\right)=3\\ &-8-4z-2-\frac{1}{3}z=3\\ &-24-12z-6-z=9\\ &-13z=39\\ &z=-3 \end{aligned}$$

Back substitute.

$$\begin{aligned} x&=-4-2(-3)=2\\ y&=2+\frac{1}{3}(-3)=1 \end{aligned}$$

Write the solution in ordered triple form: (2, 1, –3).

Practice Set 4.1

Solve the following systems of linear equations involving three variables. If an infinite number of solutions exist, find the general solution and at least one particular solution.

#1 $\begin{cases} x+2y-z=18 \\ x+y+z=27 \\ x-y-2z=-18 \end{cases}$

#2 $\begin{cases} x-2y+z=5 \\ -2x+4y-2z=2 \\ 2x+y-z=2 \end{cases}$

#3 $\begin{cases} x+y+z=3 \\ x-y+2z=14 \\ x-y-2z=-6 \end{cases}$

#4 $\begin{cases} 7x+2y-8z=33 \\ x+y+2z=13 \\ 2x-y-4z=-1 \end{cases}$

#5 $\begin{cases} x+2y-z=0 \\ 3x-y+z=6 \\ -2x-4y+2z=0 \end{cases}$

#6 $\begin{cases} x+6y-z=21 \\ -x-y+19z=-3 \\ 2x-4y+12z=56 \end{cases}$

#7 $\begin{cases} x+y+z=1 \\ 3x+y+z=2 \\ -2x+4y+8z=2 \end{cases}$

#8 $\begin{cases} x+2y+z=13 \\ 3x-y+z=9 \\ 6x-4y+z=6 \end{cases}$

<u>ANSWERS</u>

#1 (9,9,9)

#2 no solution

#3 (1,–3,5)

#4 $(5,5,{}^{3}/_{2})$

#5 infinite solutions, a general solution: $\left(\frac{12}{7}-\frac{1}{7}t,-\frac{6}{7}+\frac{4}{7}t,t\right)$, one particular solution: (1,2,5)

#6 (22,0,1)

#7 (½,¼,¼)

#8 (1,2,8)

Study Exercise 4.1

Problems

Solve the following systems of linear equations involving three variables. If there are an infinite number of solutions find the general solution and at least one particular solution.

#1
$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y \quad\quad = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

#2
$$\begin{cases} x + y - 3z = -1 \\ \quad\quad y - z = 0 \\ -x + 2y \quad = 1 \end{cases}$$

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Instruction: *Cramer's Rule*

Cramer's Rule is a method that uses determinants for solving systems of linear equations. To explain the method and how these determinants are generated, this lecture will first illustrate the solution to a system of linear equations by the addition/elimination method without simplifying the indicated products and sums of the coefficients. The intent is to demonstrate that these products and sums can be represented by determinants.

Consider system of linear equations given on the following page.

$$\begin{cases} 2x + 3y = -5 \\ 4x + y = \ \ 5 \end{cases}$$

Multiply equations so that the *y*-variable can be eliminated.

$$\begin{cases} 1\left(2x + 3y = -5\right) \\ -3\left(4x + y = \ \ 5\right) \end{cases}$$

Add the equations to eliminate the *y*:

$$\begin{cases} \ \ 1(2x) + 1(3y) = 1(-5) \\ \underline{-3(4x) - 3(1y) = -3(5)} \end{cases}$$

$$\left[1(2) - 3(4)\right]x = 1(-5) - 3(5)$$

After adding the equations and solving for *x*:

$$x = \frac{1(-5) - 3(5)}{1(2) - 3(4)}$$

Return to the original system of equations.

$$\begin{cases} 2x + 3y = -5 \\ 4x + y = \ \ 5 \end{cases}$$

Multiply equations so that the *x*-variable can be eliminated.

$$\begin{cases} -4\left(2x + 3y = -5\right) \\ \ \ 2\left(4x + y = \ \ 5\right) \end{cases}$$

Add the equations to eliminate the *x*:

$$-4(2x) - 4(3y) = -4(-5)$$
$$\underline{2(4x) + 2(1y) = 2(5)}$$
$$\left[2(1) - 4(3)\right]y = 2(5) - 4(-5)$$

After adding the equations and solving for *y*:

$$y = \frac{2(5) - 4(-5)}{2(1) - 4(3)}$$

Notice that the denominators for both *x* and *y* are the same number. This number, usually denoted as *D*, is the value of the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 2 \cdot 1 - 4 \cdot 3 = -10$$

The numerator of the *x*-value above, usually denoted, D_x, is the determinant of the coefficient matrix if the first column is replaced by the constant terms. The numerator of the *y*-value, usually denoted, D_y, is the determinant of the coefficient matrix if the second column is replaced by the constant terms.

$$D_x = \begin{vmatrix} -5 & 3 \\ 5 & 1 \end{vmatrix} = -5 \cdot 1 - 5 \cdot 3 = -5 - 15 = -20$$

$$D_y = \begin{vmatrix} 2 & -5 \\ 4 & 5 \end{vmatrix} = 2 \cdot 5 - 4 \cdot -5 = 10 + 20 = 30$$

Therefore, the values for *x* and *y* can be written in fraction form using determinants:

$$x = \frac{D_x}{D} = \frac{-20}{-10} = 2 \qquad y = \frac{D_y}{D} = \frac{30}{-10} = -3$$

Cramer's Rule establishes the relationship between determinants and solving systems of linear equations using addition/elimination methods established above. Cramer's Rule is stated below only for 2×2 systems (systems of two linear equations in two variables), but Cramer's Rule applies to all $n \times n$ systems of linear equations.

$$Given: \begin{cases} a_1 x + b_1 y = k_1 \\ a_2 x + b_2 y = k_2 \end{cases}$$

$$where$$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$$

$$if \;\; D \neq 0, then$$

$$x = \frac{D_x}{D}$$

$$y = \frac{D_y}{D}$$

If $D = 0$, then the system either has no solution or an infinite number of solutions. The system has no solution if either D_x or D_y is not equal to zero. The system is dependent and has an infinite number of solutions if both D_x and D_y equal zero.

Lecture 4.4

Example 1: $\begin{cases} 2x + y = 3 \\ 3x - 2y = 5 \end{cases}$

$$D = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -7, \quad D_x = \begin{vmatrix} 3 & 1 \\ 5 & -2 \end{vmatrix} = -11, \quad D_y = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1$$

$$x = \frac{D_x}{D} = \frac{-11}{-7} = \frac{11}{7} \qquad y = \frac{D_y}{D} = \frac{1}{-7} = -\frac{1}{7}$$

$$\left(\frac{11}{7}, -\frac{1}{7}\right)$$

Example 2: $\begin{cases} 2x - y + z = 11 \\ x + y - z = -2 \\ x + z = 3 \end{cases}$

$$D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 3, \qquad D_x = \begin{vmatrix} 11 & -1 & 1 \\ -2 & 1 & -1 \\ 3 & 0 & 1 \end{vmatrix} = 9,$$

$$D_y = \begin{vmatrix} 2 & 11 & 1 \\ 1 & -2 & -1 \\ 1 & 3 & 1 \end{vmatrix} = -15, \quad D_z = \begin{vmatrix} 2 & -1 & 11 \\ 1 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 0$$

$$x = \frac{D_x}{D} = \frac{9}{3} = 3, \quad y = \frac{D_y}{D} = \frac{-15}{3} = -5, \quad \mathrm{z} = \frac{D_z}{D} = \frac{0}{3} = 0$$

$$(3, -5, 0)$$

Example 3: $\begin{cases} x - y = 4 \\ 2x - 2y = 5 \end{cases}$

$$D = \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} = 0, \quad D_x = \begin{vmatrix} 4 & -1 \\ 5 & -2 \end{vmatrix} = -7$$

No solution.

Instruction: *Cramer's Rule*

Example 1
Solving a System of Linear Equations using Cramer's Rule

Consider the system of equations below.

$$\begin{cases} 2x - y = 11 \\ x + y = -2 \end{cases}$$

Solve the system of equations using Cramer's Rule.

Find *D*, the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} = 2(1) - 1(-1) = 2 + 1 = 3$$

Substitute the constants for the coefficients to *x* and find the determinant of the resulting 2×2 matrix, D_x.

$$D_x = \begin{vmatrix} 11 & -1 \\ -2 & 1 \end{vmatrix} = 11(1) - -2(-1) = 11 - 2 = 9$$

Substitute the constants for the coefficients to *y* and find the determinant of the resulting 2×2 matrix, D_y.

$$D_y = \begin{vmatrix} 2 & 11 \\ 1 & -2 \end{vmatrix} = 2(-2) - 1(11) = -4 - 11 = -15$$

Apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{9}{3} = 3 \qquad y = \frac{D_y}{D} = \frac{-15}{3} = -5$$

Write the solution as an ordered pair: $(3, -5)$.

Example 2
Solving a System of Linear Equations using Cramer's Rule

Consider the system of equations below.

$$\begin{cases} 2x + y - z = 5 \\ x + y - z = 0 \\ x + y = 10 \end{cases}$$

Solve the system of equations using Cramer's Rule.

Find *D*, the determinant of the coefficient matrix.

$$D = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = 1$$

Substitute the constants for the coefficients to *x* and find the determinant of the resulting 3×3 matrix, D_x.

$$D_x = \begin{vmatrix} 5 & 1 & -1 \\ 0 & 1 & -1 \\ 10 & 1 & 0 \end{vmatrix} = 5$$

Substitute the constants for the coefficients to *y* and find the determinant of the resulting 3×3 matrix, D_y.

$$D_y = \begin{vmatrix} 2 & 5 & -1 \\ 1 & 0 & -1 \\ 1 & 10 & 0 \end{vmatrix} = 5$$

Substitute the constants for the coefficients to *z* and find the determinant of the resulting 3×3 matrix, D_z.

$$D_z = \begin{vmatrix} 2 & 1 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 10 \end{vmatrix} = 10$$

Apply Cramer's Rule.

$$x = \frac{D_x}{D} = \frac{5}{1} = 5 \qquad y = \frac{D_y}{D} = \frac{5}{1} = 5 \qquad z = \frac{D_z}{D} = \frac{10}{1} = 10$$

Write the solution as an ordered triple: $(5,5,10)$.

Practice Set 4.4

Use Cramer's Rule to solve the following systems of linear equations.

#1 $\begin{cases} 5x + 7y = -1 \\ 6x + 8y = 1 \end{cases}$

#2 $\begin{cases} x + y - z = -2 \\ 2x - y + z = -5 \\ x - 2y + 3z = 4 \end{cases}$

#3 $\begin{cases} 4x - y + 3z = -3 \\ 3x + y + z = 0 \\ 2x - y + 4z = 0 \end{cases}$

#4 $\begin{cases} 2x - y + 4z = -2 \\ 3x + 2y - z = -3 \\ x + 4y + 2z = 17 \end{cases}$

#5 $\begin{cases} 2x - 3y + 4z = 10 \\ 6x - 9y + 12z = 24 \\ x + 2y - 3z = 5 \end{cases}$

ANSWERS

#1 (7.5,–5.5)

#2 $\left(-\frac{7}{3}, \frac{22}{3}, 7\right)$

#3 (–1,2,1)

#4 (–3,4,2)

#5 $D = 0$ & $D_x = 6$, therefore no solution exists.

Study Exercise 4.4

Problems

Use Cramer's Rule to solve the following systems of linear equations.

#1 $\begin{cases} 2x - 5y = 0 \\ 3x - 2y = 11 \end{cases}$

#2 $\begin{cases} 2x - y + 3z = 14 \\ 5x + 2y - z = 1 \\ x - 2y + z = 5 \end{cases}$

Lecture 4.7

College Algebra

Instruction: ***Solving Systems of Equations Using the Inverse of a Square Matrix***

To see how the inverse matrix can be used to solve a system of linear equations, consider the following system.

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = k_1 \\ a_{21}x + a_{22}y + a_{23}z = k_2 \\ a_{31}x + a_{32}y + a_{33}z = k_3 \end{cases}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

Then the system of linear equations may be written in the form of the matrix equation $AX = B$. If A is nonsingular, then its inverse can be found and the matrix equation can be solved.

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

Thus, a system of *n* equations with *n* unknowns can be solved with the matrix equation $X = A^{-1}B$ where X is the column matrix formed by the unknowns, B is the column matrix formed by the constant terms, and A is the coefficient matrix.

For example,

$$\begin{cases} 2x + y + z = 1 \\ 3x + 2y + z = 2 \\ 2x + y + 2z = -1 \end{cases}$$

Let A equal the coefficient matrix, X equal the column matrix of unknowns, and B equal the column matrix of constant terms. Then, use the method discussed above to find the inverse of A.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$AX = B$$
$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 \\ -4 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3\cdot 1 + (-1)\cdot 2 + (-1)(-1) \\ (-4)1 + 2\cdot 2 + 1(-1) \\ (-1)1 + 0\cdot 2 + 1(-1) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 - 2 + 1 \\ -4 + 4 - 1 \\ -1 + 0 - 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$x = 2, \quad y = -1, \quad z = -2$

$(2,-1,-2)$ solution to the system of linear equations written as an ordered triple

Instruction: ***Solving Systems of Linear Equations using Inverse Matrices***

Example 1
Solving a Matrix Equation

Consider the matrix equation below.

$$AX = B$$

Solve the matrix equation for X. Assume that A is a non-singular square matrix and that B has as many rows as A has columns.

Right multiply both sides of the matrix equation by the inverse of A.

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$

Recall that the product of a matrix and its inverse is the identity matrix.

$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$

Recall that the product of the identity matrix and a matrix is the matrix itself.

$$IX = A^{-1}B$$
$$X = A^{-1}B$$

Example 2
Solving a System of Linear Equations using an Inverse Matrix

Consider the system of equations below.

$$\begin{cases} 2x + y = 8 \\ x + y = 5 \end{cases}$$

Solve using an inverse matrix.

Let *A* be the coefficient matrix. Let *X* be the single column variable matrix. Let *B* be the single column constant matrix.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

Represent the given system of linear equations with a matrix equation and solve.

$$AX = B$$
$$A^{-1}AX = A^{-1}B$$
$$IX = A^{-1}B$$
$$X = A^{-1}B$$

Find A^{-1}.

$$A^{-1} = \frac{1}{2(1) - 1(1)} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

Find the solution to the given system of linear equations (the elements of *X*) by multiplying the inverse of *A* by *B*.

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 1(8) + -1(5) \\ -1(8) + 2(5) \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Write the solution as an ordered pair: $(3, 2)$.

Practice Set 4.7

Solve each of the following systems by using the inverse of the coefficient matrix.

#1 $\begin{cases} 2x - y = -8 \\ 3x + y = -2 \end{cases}$

#2 $\begin{cases} 2x - 5y = 10 \\ 4x - 5y = 15 \end{cases}$

#3 $\begin{cases} x + 2z = 7 \\ 3x + y + 5z = 19 \\ x - y + 2z = 7 \end{cases}$

#4 $\begin{cases} x + 3y + z = 5 \\ 2x - 4y + 6z = 4 \\ 2x - 3y - z = -2 \end{cases}$

#5 The points (1,9), (4,36), and (–3,1) lie on the curve described by the function $y = ax^2 + bx + c$. Find the values of a, b, and c.

#6 The points (1,12), (0,3), and (–2, –3) lie on the curve described by the function $y = ax^2 + bx + c$. Find the values of a, b, and c.

#7 The points (1,6), (0,2), and (–3,2) lie on the curve described by the function $y = ax^2 + bx + c$. Find the values of a, b, and c.

#8 The points (4, –8), (0,12), and (–2,10) lie on the curve described by the function $y = ax^2 + bx + c$. Find the values of a, b, and c.

ANSWERS

#1 (–2,4)
#2 ($^5/_2$, –1)
#3 (3,0,2)
#4 (1,1,1)
#5 $y = x^2 + 4x + 4$
#6 $y = 2x^2 + 7x + 3$
#7 $y = x^2 + 3x + 2$
#8 $y = -x^2 - x + 12$

Study Exercise 4.7

Problems

Solve the following system of equations using inverse matrices.

#1 Solve $\begin{cases} 2x - y = 1 \\ 3x + y = 6 \end{cases}$ using an inverse matrix.

#2 Solve $\begin{cases} x + y + 2z = 11 \\ 2x + 4y - 3z = 23 \\ 3x + 6y - 5z = 34 \end{cases}$, using an inverse matrix.