Lecture 4.10

College Algebra

Instruction: *Partial Fraction Decomposition*

In chapter four, we have learned several strategies for solving systems of linear equations. One use for solving systems of linear equations occurs in a process called partial fraction decomposition.

There are situations in mathematics when it is helpful to write an algebraic fraction as the sum of simpler fractions, that is, to decompose the fraction into partial fractions whose sum is the original. Partial fraction decomposition is a two-step process. The first step produces the form of the answer, that is, the form of the addends called partial fractions. The second step finds the answer, that is, the actual addends.

We will start by addressing the first step. Suppose we have the fraction below.

$$\frac{3x+5}{x^2+x-12}$$

Suppose further that we want to write this fraction as the sum of simpler fractions. There are two preliminary considerations. First, we consider whether the degree of the numerator is less than the degree of the denominator. If it is, fine. If not, long divide and perform the remainder of the process (described hereafter) on the remainder divided by the denominator. Second, we consider whether the denominator is written as a product of prime factors. If it is, fine. If not, factor it into the lowest degreed factors possible. (The problems we work will always be easily factorable into linear or quadratic factors).

Now, let us return to our example above. Note that the degree of the numerator is smaller than the degree of the denominator. Accordingly, we may proceed without having to long-divide. Note further that the denominator is not written as a product of prime factors. Consequently, we should factor before proceeding further.

$$\frac{3x+5}{(x+4)(x-3)}$$

We are ready now, to find the form of the addends of the partial fractions.

The factors of the denominator will fall into one of four categories: linear non-repeated, linear repeated, quadratic non-repeated, and quadratic repeated. In our example, $(x-3)$ is linear because *inside* the parentheses the biggest exponent applied to a variable is one. Moreover, $(x-3)$ is non-repeated because *outside* the parentheses the biggest exponent is one, i.e., the factor only appears once. The same is true for the factor $(x+4)$.

At this point, let us break off from this example for a while and discuss classifying factors in the denominator. The left column of the table on the following page presents factors; the right column presents their classification.

Factors	**Classification**
$2x - 3$	linear, non-repeated
$(2x - 3)^2$	linear, repeated
$x^2 + 4$	quadratic, non-repeated
$(x^2 + x + 1)^2$	quadratic, repeated
$(10 - x)$	linear, non-repeated

The next table divides the information into three columns. The left column presents composite polynomials. The middle column presents the composite polynomials as a product of prime factors. The right column classifies the prime factors found in the middle column.

Polynomials	**Prime Factors**	**Classification**
$x^2 - 4$	$(x + 2)(x - 4)$	both are linear, non-repeated
$x^3 - 8$	$(x - 2)(x^2 + 2x + 4)$	left factor is linear, non-repeated & right factor is quadratic, non-repeated
$x^4 + 8x^2 + 16$	$(x^2 + 4)^2$	quadratic, repeated

The only factor that may cause the reader mild consternation is the innocent-looking x^2. Since x^2 is recognizable as the parent quadratic function, one might think of it as a quadratic factor, but it is a repeated linear factor. Remember that x is the parent linear function and that x^2 is simply x repeated.

Now that we are adept at classifying factors, we can turn our attention to writing the form of the addends of the partial fractions. There are four forms outlined below.

I. If a factor is linear, non-repeated like $(x - 3)$, it will produce an addend that will look like $\frac{A}{x-3}$ where A is a constant to be determined later.

II. If a factor is linear, repeated like $(x - 3)^3$, it will produce multiple addends, one for each stage of repetition of the factor. These addends will look like $\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{(x-3)^3}$ where A, B, and C are constants to be determined later.

III. If a factor is quadratic, non-repeated like $(x^2 + x + 1)$, it will produce an addend that will look like $\frac{Ax+B}{x^2+x+1}$ where A and B are constants to be determined later.

IV. If a factor is a quadratic, repeated like $(x^2 + 4)^2$, it will produce multiple addends, one for each stage of repetition of the factor. These addends will look like $\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$ where A, B, C, and D are constants to be determined later.

Keep in mind that the outline above refers not to finding the specific partial fractions (the ultimate answer). Instead, it refers only to finding the *form* of the partial fractions (the form of the ultimate answer). Returning to our original example, we can now write the form of its partial fractions as shown here.

$$\frac{3x+5}{x^2+x-12}=\frac{3x+5}{(x-3)(x+4)}=\frac{A}{x-3}+\frac{B}{x+4}$$

We choose *B* to represent the constant over the second factor to distinguish it from *A*. Traditionally, the symbols used to represent the constants progress through the alphabet in alphabetical order.

At this point, let us practice writing the form of the addends of a partial fraction decomposition. The left column of the table below presents an algebraic fraction. The right column presents the form of the decomposition.

Fraction	Form of Decomposition
$\frac{x^2+5}{(x-3)(2x+7)(5x-3)}$	$\frac{A}{x-3}+\frac{B}{2x+7}+\frac{C}{5x-3}$
$\frac{12}{(x^2+9)(2x+5)}$	$\frac{Ax+B}{x^2+9}+\frac{C}{2x+5}$
$\frac{x^2+5x+1}{(x^2+2x+5)^2(x-1)^2}$	$\frac{Ax+B}{x^2+2x+5}+\frac{Cx+D}{(x^2+2x+5)^2}+\frac{E}{x-1}+\frac{F}{(x-1)^2}$
$\frac{x^2+12x+3}{(x+1)^3(4x-3)^2}$	$\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}+\frac{D}{4x-3}+\frac{E}{(4x-3)^2}$
$\frac{10-x}{(5x+4)(x+2)^2(x^2+x+1)}$	$\frac{A}{5x+4}+\frac{B}{x+2}+\frac{C}{(x+2)^2}+\frac{Dx+E}{x^2+x+1}$
$\frac{8}{x^2(x+5)}$	$\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+5}$
$\frac{x+2}{x^2+7x+12}$	$\frac{A}{x+3}+\frac{B}{x+4}$
$\frac{3x-1}{x(x^2+7x+1)}$	$\frac{A}{x}+\frac{Bx+C}{x^2+7x+1}$

The reader should note that the numerator of each fraction plays no part whatsoever in the form of the decomposition. The numerator does not become important until the process for finding the actual addends (not just their forms) proceeds. We will turn to this process now.

After writing the form of the addends, finding the constants represented by *A*, *B*, *C*, and so on completes each partial fraction. The technique for finding the constants has six steps given here.

1. Set the algebraic fraction equal to the forms of its partial fractions.

2. Multiply both sides by the denominator of the decomposable fraction.

3. Write the side of the equation with the undetermined constants as a polynomial in standard form.

4. Equate the corresponding coefficients of the powers of the variable in the two polynomials.

5. Solve the resulting system of linear equations to determine the unknown constants.

6. Write the partial fractions.

Let us now apply this process to our example and carry out the six steps.

1. $$\frac{3x+5}{(x-3)(x+4)} = \frac{A}{x-3} + \frac{B}{x+4}$$

2. $$3x+5 = A(x+4) + B(x-3)$$

3. $$3x+5 = Ax + 4A + Bx - 3B$$
$$3x+5 = Ax + Bx + 4A - 3B$$
$$3x+5 = (A+B)x + 4A - 3B$$

4. $$A + B = 3$$
$$4A - 3B = 5$$

5. $$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 3/7 & 1/7 \\ 4/7 & -1/7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$
$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

6. $$\frac{3x+5}{(x-3)(x+4)} = \frac{2}{x-3} + \frac{1}{x+4}$$

The reader will notice that the system of linear equations was solved using the inverse matrix. Any other method for solving the system of equations will work including substitution, elimination by addition/multiplication, Gaussian elimination, Gauss-Jordan elimination, even Cramer's Rule. Adding the partial fractions to see if the sum equals the original fraction checks the answer.

Instruction: *Partial Fraction Decomposition, More Examples*

In this assignment, students will find the sum of partial fractions that equal an algebraic fraction. This process, called partial fraction decomposition, is a two-step process. The first step gives the form of the answer, that is, the form of the addends called partial fractions. The second step finds the answer, that is, the actual addends.

Suppose we have the fraction below.

$$\frac{26x-17}{(2x-3)^2(x+4)}$$

If the degree of the numerator exceeded the degree of the denominator, we would long divide first and then decompose the remainder divided by the denominator. Since the degree of the denominator exceeds the degree of the numerator, we may proceed as it is. Moreover, the denominator is factored saving us the trouble. The first factor, $2x - 3$, is a linear, repeated factor. It is repeated because there are two of them as indicated by the exponent outside the parentheses. The second factor is a linear, non-repeated factor. Linear factors produce addends with a single constant in the numerator. Repeated factors produce multiple addends, one for each stage of the repetition. Accordingly, the fraction will produce the following sum:

$$\frac{26x-17}{(2x-3)^2(x+4)} = \frac{A}{2x-3} + \frac{B}{(2x-3)^2} + \frac{C}{x+4}$$

We have written the form of the addends, but now we must find the constants A, B, and C with the technique described in the box below.

1. Set the algebraic fraction equal to the forms of its partial fractions.

2. Multiply both sides by the denominator of the decomposable fraction.

3. Write the side of the equation with the undetermined constants as a polynomial in standard form.

4. Equate the corresponding coefficients of the powers of the variable in the two polynomials.

5. Solve the resulting system of linear equations to determine the unknown constants.

6. Write the partial fractions.

Let us now apply this process to our example and carry out the six steps.

1. $$\frac{26x-17}{(2x-3)^2(x+4)}=\frac{A}{2x-3}+\frac{B}{(2x-3)^2}+\frac{C}{x+4}$$

2. $$26x-17=A(2x-3)(x+4)+B(x+4)+C(2x-3)^2$$

3. $$26x-17=A(2x^2+5x-12)+Bx+4B+C(4x^2-12x+9)$$
$$26x-17=2Ax^2+5Ax-12A+Bx+4B+4Cx^2-12Cx+9C$$
$$26x-17=2Ax^2+4Cx^2+5Ax+Bx-12Cx-12A+4B+9C$$
$$26x-17=(2A+4C)x^2+(5A+B-12C)x+(-12A+4B+9C)$$

4. $$2A+4C=0$$
$$5A+B-12C=26$$
$$-12A+4B+9C=-17$$

5. $$\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}2&0&4\\5&1&-12\\-12&4&9\end{bmatrix}^{-1}\cdot\begin{bmatrix}0\\26\\-17\end{bmatrix}$$
$$\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}57/242&8/121&-2/121\\9/22&3/11&2/11\\16/121&-4/121&1/121\end{bmatrix}\cdot\begin{bmatrix}0\\26\\-17\end{bmatrix}$$
$$\begin{bmatrix}A\\B\\C\end{bmatrix}=\begin{bmatrix}2\\4\\-1\end{bmatrix}$$

6. $$\frac{26x-17}{(2x-3)^2(x+4)}=\frac{2}{2x-3}+\frac{4}{(2x-3)^2}-\frac{1}{x+4}$$

In step five, the example uses an inverse matrix to solve the system of linear equations established in step four. Any other method is sufficient for solving the system of equations including substitution, elimination by addition/multiplication, Gaussian elimination, Gauss-Jordan elimination, even Cramer's Rule. Adding the partial fractions to see if the sum equals the original fraction checks the answer.

The next example reveals how to approach an algebraic fraction that has a quadratic factor in the denominator.

Find the partial fraction decomposition of $\dfrac{5x^2+15x+4}{(x^2+2x+3)(x+5)}$.

Start by acquiring the form of the answer: $\dfrac{5x^2+15x+4}{(x^2+2x+3)(x+5)} = \dfrac{Ax+B}{x^2+2x+3} + \dfrac{C}{x+5}$.

Denominators with quadratic factors create addends with a linear factor, $Ax + B$, in the numerator.

Multiply both sides by the original fraction's denominator:

$$(x^2+2x+3)(x+5)\cdot\left[\frac{5x^2+15x+4}{(x^2+2x+3)(x+5)}\right] = \left[\frac{Ax+B}{x^2+2x+3}+\frac{C}{x+5}\right]\cdot(x^2+2x+3)(x+5)$$

$$5x^2+15x+4 = (Ax+B)(x+5)+C(x^2+2x+3)$$

Express the right side of the equation as a polynomial:

$$5x^2+15x+4 = (Ax+B)(x+5)+C(x^2+2x+3)$$
$$5x^2+15x+4 = Ax^2+5Ax+Bx+5B+Cx^2+2Cx+3C$$
$$5x^2+15x+4 = Ax^2+Cx^2+5Ax+Bx+2Cx+5B+3C$$
$$5x^2+15x+4 = (A+C)x^2+(5A+B+2C)x+(5B+3C)$$

Equate the corresponding coefficients to the two polynomials in the equation:

$$A+C=5$$
$$5A+B+2C=15$$
$$5B+3C=4$$

Solve the system of linear equations for the constants. This example uses Cramer's rule.

$$D = \begin{vmatrix} 1 & 0 & 1 \\ 5 & 1 & 2 \\ 0 & 5 & 3 \end{vmatrix} = 18$$

$$D_A = \begin{vmatrix} 5 & 0 & 1 \\ 15 & 1 & 2 \\ 4 & 5 & 3 \end{vmatrix} = 36 \qquad A = \frac{D_A}{D} = \frac{36}{18} = 2$$

$$D_B = \begin{vmatrix} 1 & 5 & 1 \\ 5 & 15 & 2 \\ 0 & 4 & 3 \end{vmatrix} = -18 \qquad B = \frac{D_B}{D} = \frac{-18}{18} = -1$$

$$D_C = \begin{vmatrix} 1 & 0 & 5 \\ 5 & 1 & 15 \\ 0 & 5 & 4 \end{vmatrix} = 54 \qquad C = \frac{D_C}{D} = \frac{54}{18} = 3$$

Write the partial fraction decomposition:

$$\frac{5x^2 + 15x + 4}{(x^2 + 2x + 3)(x + 5)} = \frac{2x - 1}{x^2 + 2x + 3} + \frac{3}{x + 5}.$$

Instruction: *Partial Fraction Decomposition*

Example 1
Decomposing a Fraction to Partial Fractions

Decompose the fraction: $\dfrac{x+4}{x^2-3x-10}$.

Factor the denominator.

$$\frac{x+4}{(x-5)(x+2)}.$$

Surmise that the "partial fractions" must have had x – 5 and x + 2 as their respective denominators in order to end up with (x – 5)(x + 2) as the common denominator. Assume the numerators are constants. Place an unknown above one "partial fraction" to represent its numerator. Place another unknown in the numerator of the other "partial fraction" as shown below.

$$\frac{x+4}{(x-5)(x+2)}=\frac{A}{x-5}+\frac{B}{x+2}$$

Multiply both sides of the equation by the common denominator.

$$x+4=A(x+2)+B(x-5)$$

The resulting equation has the linear expression x + 4 equal to an expression that must also be linear (indeed it must also be x + 4) if the two sides of the equation are truly equal. Simplify the right side of the equation.

$$x+4=A(x+2)+B(x-5)$$
$$x+4=Ax+2A+Bx-5B$$
$$x+4=Ax+Bx+2A-5B$$
$$x+4=(A+B)x+2A-5B$$

A + B must be equal 1 because these are two equal linear factors and the linear coefficient on the left is 1, so the linear coefficient on the right (A + B) must also be 1. Similarly, the constants are equal so 2A – 5B must equal 4. Write a system of equations.

$$A+B=1$$
$$2A-5B=4$$

Solve this system for A and B.

$$\begin{bmatrix} A \\ B \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix}^{-1}\cdot\begin{bmatrix} 1 \\ 4 \end{bmatrix}=\begin{bmatrix} 9/7 \\ -2/7 \end{bmatrix}$$

A and *B* represent the numerators of the partial fractions. Write the answer.

$$\frac{x+4}{(x-5)(x+2)} = \frac{9/7}{x-5} - \frac{2/7}{x+2}$$

Or, equivalently:

$$\frac{x+4}{(x-5)(x+2)} = \frac{9}{7(x-5)} - \frac{2}{7(x+2)}.$$

Example 2

Setting up an Equations whose Unknowns are in the Numerators of the Partial Fractions

Consider the fraction: $\frac{x^2+2}{(x+1)^3(x^2+1)}$. Write the equation whose unknowns are the coefficients and constants of the numerators of the partial fractions.

Recognize the repeated linear factor and the quadratic factor in the denominator. Write an equation setting the given fraction equal to the sum of four "partial fractions" whose denominators are the quadratic factor, the linear factor, the linear factor repeated, and the linear factor repeated three times. Assume constant numerators for fractions with linear factors in the denominator and linear numerators for fractions with quadratic factors in the denominator.

$$\frac{x^2+2}{(x+1)^3(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}.$$

Practice Set 4.10

Write the partial fraction decomposition for the following algebraic fractions.

#1 $\dfrac{x+6}{x^2-4}$

#2 $\dfrac{5x-3}{2x^2-9x+9}$

#3 $\dfrac{4x+2}{x^2+2x}$

#4 $\dfrac{4x^2-3x-4}{x^3+x^2-2x}$

#5 $\dfrac{-x^2-5}{(x^2+x+1)(2x-1)}$

#6 $\dfrac{15x^3+19x^2+15x+6}{2x^4+3x^3+2x^2+3x}$

#7 $\dfrac{2x^2+16}{x^3+8}$

#8 $\dfrac{x^3-8}{x^5-3x^4+3x^3-x^2}$

ANSWERS

#1 $\dfrac{x+6}{x^2-4}=\dfrac{2}{x-2}-\dfrac{1}{x+2}$

#2 $\dfrac{5x-3}{2x^2-9x+9}=\dfrac{4}{x-3}-\dfrac{3}{2x-3}$

#3 $\dfrac{4x+2}{x^2+2x}=\dfrac{1}{x}+\dfrac{3}{x+2}$

#4 $\dfrac{4x^2-3x-4}{x^3+x^2-2x}=\dfrac{2}{x}-\dfrac{1}{x-1}+\dfrac{3}{x+2}$

#5 $\dfrac{-x^2-5}{(x^2+x+1)(2x-1)}=\dfrac{x+2}{x^2+x+1}-\dfrac{3}{2x-1}$

#6 $\dfrac{15x^3+19x^2+15x+6}{2x^4+3x^3+2x^2+3x}=\dfrac{2}{x}+\dfrac{5}{2x+3}+\dfrac{3x+2}{x^2+1}$

#7 $\dfrac{2x^2+16}{x^3+8}=\dfrac{2}{x+2}+\dfrac{4}{x^2-2x+4}$

#8 $\dfrac{x^3-8}{x^5-3x^4+3x^3-x^2}=\dfrac{8}{x^2}+\dfrac{24}{x}-\dfrac{7}{(x-1)^3}+\dfrac{17}{(x-1)^2}-\dfrac{24}{x}$

Practice Set 4.10_Supplemental

Write the partial fraction decomposition for the following algebraic fractions. When necessary, use synthetic division to factor the denominators.

#1 $$\frac{5x+6}{2x^3+3x^2-17x-30}$$

#2 $$\frac{8x^2+7x+11}{x^3+x^2+5x+5}$$

#3 $$\frac{5x^3+16x+5}{x^4-3x^3+x^2+4}$$

#4 $$\frac{2x^3+11x^2+9x+5}{\left(x^2+5x+2\right)^2}$$

#5 $$\frac{4x^3-2x^2-5x-12}{x^4-4x^3}$$

<u>ANSWERS</u>

#1 $$\frac{5x+6}{2x^3+3x^2-17x-30}=\frac{4/5}{x+2}-\frac{26/11}{2x+5}+\frac{21/55}{x-3}=\frac{44}{55(x+2)}-\frac{130}{55(2x+5)}+\frac{21}{55(x-3)}$$

#2 $$\frac{8x^2+7x+11}{x^3+x^2+5x+5}=\frac{6x+1}{x^2+5}+\frac{2}{x+1}$$

#3 $$\frac{5x^3+16x+5}{x^4-3x^3+x^2+4}=\frac{2x}{x^2+x+1}+\frac{3}{x-2}+\frac{11}{(x-2)^2}$$

#4 $$\frac{2x^3+11x^2+9x+5}{\left(x^2+5x+2\right)^2}=\frac{2x+1}{x^2+5x+2}+\frac{3}{\left(x^2+5x+2\right)^2}$$

#5 $$\frac{4x^3-2x^2-5x-12}{x^4-4x^3}=\frac{3}{x^3}+\frac{2}{x^2}+\frac{1}{x}+\frac{3}{x-4}$$

Study Exercise 4.10

Problems

Write the partial fraction decomposition for the following algebraic fractions.

#1 $$\frac{x+14}{(x-4)(x+2)}$$

#2 $$\frac{3x^2+17x+14}{(x-2)(x^2+2x+4)}$$