College Algebra

Instruction: *Solving Simple Logarithmic Equations*

Consider the equation: $\log_4 x = 2$. Remember, a logarithm is an exponent. The logarithm, $\log_4 x$, is the exponent applied to the base 4, and it equals 2. The given equation can, therefore, be written in exponential form as $4^2 = x$. Thus, $x = 16$ and the equation is solved. Another example:

$$\log_2\left(x+\frac{1}{2}\right)=-2$$
$$2^{-2}=x+\frac{1}{2}$$
$$\frac{1}{4}=x+\frac{1}{2}$$
$$\frac{1}{4}-\frac{1}{2}=x$$
$$\frac{1}{4}-\frac{2}{4}=x$$
$$-\frac{1}{4}=x$$

Consider the equation: $\log_3 27 = x + 1$. Since a logarithm is an exponent and this logarithm equals $x + 1$, the variable is an exponent. The given equation can, therefore, be written in exponential form as $3^{x+1} = 27$. To solve the equation, express the right side as a power of three: $3^{x+1} = 3^3$. Since the bases are alike, the exponents must be equal: $x + 1 = 3$. Thus, $x = 2$. Another example:

$$\log_5 \frac{1}{625}=2x-8$$
$$5^{2x-8}=\frac{1}{625}$$
$$5^{2x-8}=\frac{1}{5^4}$$
Write $^1/_{625}$ as an exponential expression with 5 as its base.
$$5^{2x-8}=5^{-4}$$
$$2x-8=-4$$
Since the bases are equal, the exponents must be equal.
$$2x=4$$
$$x=2$$

Instruction: ***Introducing Logarithmic Equations***

Example 1
Solving Logarithmic Equations

Solve $\log_3(x+7)=4$.

Exponentiate and solve the resulting linear equation.

$$\begin{aligned}
&\log_3(x+7)=4\\
&x+7=3^4\\
&x+7=81\\
&x=81-7\\
&x=74
\end{aligned}$$

Example 2
Solving Logarithmic Equations

Solve $\log(x^2)=2$.

Exponentiate and solve the resulting quadratic equation.

$$\begin{aligned}
&\log(x^2)=2\\
&x^2=10^2\\
&x^2=100\\
&\sqrt{x^2}=\pm\sqrt{100}\\
&x=\pm 10
\end{aligned}$$

Practice Set 3.8

Solve the following equations for the unknown. Remember the definition of logarithm given in the following box.

For all positive numbers b, where $b \neq 1$, $b^x = a$ is equivalent to $x = \log_b a$.

In words, a logarithm is an exponent, and $\log_b a$ is the exponent to which b must be raised in order to obtain a. The number b is the *base* of the logarithm, and a is the *argument* of the expression $\log_b a$. The value of a will always be positive.

#1 $\log_3(x+7) = 3$

#2 $\log_{\frac{4}{5}}\left(\frac{x}{16}\right) = -2$

#3 $\log_4 32 = x + 1$

#4 $\log_x 36 = 2$

#5 $\log_2(2x+5) = 3$

#6 $\log_7(5x+14) = 2$

#7 $\log_4(x^2 - x + 4) = 2$

#8 $\log(x^2 + 900) = 3$

ANSWERS

#1) $x = 20$ #2) $x = 25$

#3) $x = 1.5$ #4) $x = 6$

#5) $x = 1.5$ #6) $x = 7$

#7) $x = -3$ or $x = 4$ #8) $x = -10$ or $x = 10$

Study Exercise 3.8

Problems

Solve the following equations for the exact values of *x*.

#1 $\log_{\frac{1}{2}} x = -3$

#2 $\log_2 (3x + 14) = 3$

#3 $-5 = \log_4 \left(\frac{x}{4096} \right)$

#4 $\log_3 \frac{1}{81} = x$

College Algebra

Instruction: *Solving Logarithmic Equations Using Properties of Logarithms*

The following five-step rubric may help students solve logarithmic equations with variables as arguments. Consider the equation: $\log(x-1)+\log(x-4)+5=6.$

The first step requires that the logarithms be written as a single logarithm. The logarithm should also be isolated:

$$\begin{aligned}\log(x-1)+\log(x-4)+5&=6\\ \log(x-1)+\log(x-4)&=6-5\\ \log(x-1)+\log(x-4)&=1\\ \log(x-1)(x-4)&=1.\end{aligned}$$

Isolate the logs by subtracting five from both sides.

Use the property, log$x$ + log$y$ = log(xy) to write the sum of logarithms as a single logarithm.

Now, both sides of the equation should be exponentiated, i.e., both sides of the equation should be written as exponents. Of course, each side will require a base in order to be written as exponents and the same base must be used on both sides. Which base should be used? Since the logarithm in the equation is a common logarithm (a logarithm to the base ten), base ten should be used:

$$\begin{aligned}\log(x-1)(x-4)&=1\\ 10^{\log(x-1)(x-4)}&=10^1.\end{aligned}$$

The third step requires the student to use the property of exponents that states $b^{\log_b x}=x.$ Using this property, the left side reduces to the argument:

$$\begin{aligned}10^{\log(x-1)(x-4)}&=10^1\\ (x-1)(x-4)&=10^1.\end{aligned}$$

The final step requires that the resulting equation be solved algebraically:

$$\begin{aligned}(x-1)(x-4)&=10^1\\ x^2-5x+4&=10\\ x^2-5x-6&=0\\ (x-6)(x+1)&=0\\ x-6=0,\ x+1&=0\\ x=6,\qquad x&\neq-1\end{aligned}$$

The final step requires that the solutions be checked in the original equation. The solution, –1, must be rejected since it does not check in the original equation. It does not check because the argument of a logarithm must be a positive number to be defined.

Lecture 3.9

To review, read the rubric below:

1 Write multiple logarithms as a single logarithm. Isolate the logarithms if possible.
2 Exponentiate both sides.
3 Apply the property of logarithms that states $b^{\log_b x} = x$.
4 Solve the resulting equation.
5 Check the solution to be sure the arguments in the original equation are positive.

The following examples solve logarithmic equations using the above rubric.

Example 2:

$$\log_2 x - \log_2 (x-1) = \log_2 3$$
$$\log_2\left(\frac{x}{x-1}\right) = \log_2 3$$
$$2^{\log\left(\frac{x}{x-1}\right)} = 2^{\log 3}$$
$$\frac{x}{x-1} = 3$$
$$x = 3(x-1)$$
$$x = 3x - 3$$
$$3 = 2x$$
$$\frac{3}{2} = x$$

The answer checks.

Example 3:

$$\ln(x^2 - x - 6) - \ln(x+2) = 2$$
$$\ln\left(\frac{x^2 - x - 6}{x+2}\right) = 2$$
$$\ln\left(\frac{(x+2)(x-3)}{x+2}\right) = 2$$
$$\ln(x-3) = 2$$
$$e^{\ln(x-3)} = e^2$$
$$x - 3 = e^2$$
$$x = e^2 + 3 \longleftarrow \text{Exact solution.}$$
$$x \approx 7.389 + 3$$
$$x \approx 10.389 \longleftarrow \text{Approximation of solution.}$$

The answer checks.

Lecture 3.9

Example 4.

$$5\ln x - 8 = 0$$
$$5\ln x = 8$$
$$\ln x = \frac{8}{5}$$
$$e^{\ln x} = e^{\frac{8}{5}}$$
$$x = e^{\frac{8}{5}}$$ ← Exact solution.
$$x \approx 4.953$$ ← Approximation of solution.

The answer checks.

Example 5:

$$\log_4 \sqrt{x^2 - 24} = \frac{3}{2}$$
$$\log_4 \left(x^2 - 24\right)^{\frac{1}{2}} = \frac{3}{2}$$
$$\frac{1}{2}\log_4\left(x^2 - 24\right) = \frac{3}{2}$$
$$2 \cdot \left[\frac{1}{2}\log_4\left(x^2 - 24\right)\right] = 2 \cdot \left[\frac{3}{2}\right]$$
$$\log_4\left(x^2 - 24\right) = 3$$
$$4^{\log_4\left(x^2 - 24\right)} = 4^3$$
$$\left(x^2 - 24\right) = 4^3$$
$$x^2 - 24 = 64$$
$$x^2 = 88$$
$$\sqrt{x^2} = \sqrt{88}$$
$$x = \pm\sqrt{4 \cdot 22}$$
$$x = \pm 2\sqrt{22}$$ ← Exact solution.
$$x \approx \pm 9.38$$ ← Approximation of solution.

Both answers check in example 5. In some cases, only one of two answers will check.

Instruction: *Logarithmic Equations*

Example 1
Solving Logarithmic Equations

Solve $\log_2(16x) - 2 = -\log_2(x)$.

Transform the equation such that all logarithmic expressions are isolated on one side.

$$\log_2(16x) - 2 = -\log_2(x)$$
$$\log_2(16x) + \log_2(x) = 2$$

Apply logarithmic properties in order to create a single logarithm.

$$\log_2(16x) + \log_2(x) = 2$$
$$\log_2(16x \cdot x) = 2$$
$$\log_2(16x^2) = 2$$

Exponentiate and solve the resulting quadratic equation.

$$\log_2(16x^2) = 2$$
$$16x^2 = 2^2$$
$$16x^2 = 4$$
$$x^2 = \frac{4}{16}$$
$$\sqrt{x^2} = \sqrt{\frac{1}{4}}$$
$$x = \pm\frac{1}{2}$$

Recall that the argument of a logarithm must be a positive number. Discard $-1/2$ since it creates undefined expressions in the original equation.

Example 2
Solving Logarithmic Equations

Solve $\log_3(2-x) = 2 - \log_3(4+x)$.

Transform the equation such that all logarithmic expressions are isolated on one side.

$$\log_3(2-x) = 2 - \log_3(4+x)$$
$$\log_3(2-x) + \log_3(4+x) = 2$$

Apply logarithmic properties in order to create a single logarithm.

$$\log_3(2-x) + \log_3(4+x) = 2$$
$$\log_3\left[(2-x)(4+x)\right] = 2$$
$$\log_3\left(8-2x-x^2\right) = 2$$

Exponentiate and solve the resulting quadratic equation.

$$\log_3\left(8-2x-x^2\right) = 2$$
$$8-2x-x^2 = 3^2$$
$$-x^2-2x+8 = 9$$
$$-x^2-2x-1 = 0$$
$$x^2+2x+1 = 0$$
$$(x+1)(x+1) = 0$$
$$x = -1$$

Example 3
Solving Logarithmic Equations

Solve $\log_6(1-x)+\log_6(-x)=1$.

Apply logarithmic properties in order to create a single logarithm.

$$\log_6(1-x)+\log_6(-x)=1$$
$$\log_6\left[(1-x)\cdot -x\right]=1$$
$$\log_6\left(-x+x^2\right)=1$$

Exponentiate and solve the resulting quadratic equation.

$$\log_6\left(-x+x^2\right)=1$$
$$-x+x^2=6^1$$
$$x^2-x=6$$
$$x^2-x-6=0$$
$$(x-3)(x+2)=0$$
$$x-3=0, \quad x+2=0$$
$$x=3, \qquad x=-2$$

Recall that the argument of a logarithm must be a positive number. Discard 3 since it creates undefined expressions in the original equation.

Example 4
Solving Logarithmic Equations

Solve $\log_5(5x)-\log_5(x-2)=3$.

Apply logarithmic properties in order to create a single logarithm.

$$\log_5(5x)-\log_5(x-2)=3$$
$$\log_5\left(\frac{5x}{x-2}\right)=3$$

Exponentiate and solve the resulting equation.

$$\log_5\left(\frac{5x}{x-2}\right)=3$$
$$\frac{5x}{x-2}=5^3$$
$$\frac{5x}{x-2}=125$$
$$5x=125(x-2)$$
$$5x=125x-250$$
$$5x-125x=-250$$
$$-120x=-250$$
$$x=\frac{-250}{-120}$$
$$x=\frac{25}{12}$$

Example 5
Solving Logarithmic Equations

Solve $3\ln(x+3)-26=1$. Leave all solutions exact.

Transform the equation such that all logarithmic expressions are isolated on one side.

$$3\ln(x+3)-26=1$$
$$3\ln(x+3)=27$$
$$\ln(x+3)=9$$

Exponentiate and solve the resulting linear equation.

$$\ln(x+3)=9$$
$$x+3=e^{9}$$
$$x=e^{9}-3$$

Example 6
Solving Logarithmic Equations

Solve $8\cdot\log(5x+3)+7=10$. Round the solution to the nearest thousandth.

Transform the equation such that all logarithmic expressions are isolated on one side.

$$8\cdot\log(5x+3)+7=10$$
$$8\cdot\log(5x+3)=3$$
$$\log(5x+3)=3/8$$

Exponentiate and solve the resulting linear equation.

$$\log(5x+3)=3/8$$
$$5x+3=10^{\frac{3}{8}}$$
$$5x=10^{\frac{3}{8}}-3$$
$$x=\frac{10^{\frac{3}{8}}-3}{5}\approx-0.126$$

Practice Set 3.9

Solve the following equations for x. Remember that the argument of a logarithm must be positive.

#1 $\log_2(x) + \log_2(x) = \log_2(81)$

#2 $\ln(2x+18) = \ln(5x+3)$

#3 $\log_3 x - \log_3(x-8) = 2$

#4 $\log_2(7-x) + \log_2(-x) = 3$

#5 $\log(3x) = \log(9-x)$

#6 $\log_2(x-6) + \log_2(x+6) = 6$

Solve the following equations for x. Round answers to the nearest thousandths.

#7 $4\log_2 x - 3 = 0$

#8 $5\ln(2x) + 1 = 0$

ANSWERS

#1) $x = 9$	#2) $x = 5$
#3) $x = 9$	#4) $x = -1$
#5) $x = {}^{9}/_{4}$	#6) $x = 10$
#7) $x \approx 1.682$	#8) $x \approx 0.409$

Study Exercise 3.9

Problems

Solve the following equations for *x*. Leave all answers exact.

#1 $\log(x)+\log(x-3)=1$

#2 $\log_2(x+5)-\log_2(x-2)=3$

#3 $\log_4(x)+\log_4(x-1)=\frac{1}{2}$

#4 $25\ln(x+1)+1925=2000$