Lecture 3.1

College Algebra

Instruction: *Introducing Exponential Functions*

Chapter three of this course examines exponential functions. Initially, we will regard exponential functions as functions of the form given in the box below.

> Exponential functions are functions of the form $y(x) = a \cdot (b)^{kx}$ where *a*, *b*, and *k* are positive real numbers, $b \neq 1$. Alternatively, exponential functions are functions of the form $y(x) = a \cdot (1 + r)^{kx}$ where *a* and *k* are positive real numbers and *r* is a non-zero real number.

As an aside to this lecture, we will address the restrictions on *a* and *k* with a note here. The coefficients *a* and *k* are designated as positive real numbers only for our discussion. Reflections of the function $y(x) = a \cdot (b)^{kx}$ are also exponential functions. For example, $-y(x) = -a \cdot (b)^{kx}$, is a reflection of *y*(*x*) over the *x*-axis, and is an exponential function. Similarly, $y(-x) = a(b)^{-kx}$, is a reflection of *y*(*x*) over the *y*-axis, and is an exponential function.

For functions of the form $y(x) = a \cdot (b)^{kx}$, the coefficient *a* represents an **initial amount** of the function. The initial amount will equal the *y*-value of the *y*-intercept as long as the function has not undergone any vertical or horizontal shifts. Thus, the *y*-intercept of a non-translated exponential function is (0,*a*). If a real number *c* is added to an exponential function, translating the function up or down (shifting vertically), then the *y*-intercept of the function is (0,*a* + *c*). Such a value, *c*, that translates an exponential function up or down is called the asymptotic value, and when such a value occurs as in $y(x) = a \cdot (b)^{kx} + c$ or $y(x) = c + a \cdot (b)^{kx}$, then the exponential function will have a horizontal asymptote at $y = c$. The function will approach the horizontal asymptote as *x* values approach infinity if $b < 1$ or as the *x*-values approach negative infinity if $b > 1$.

The base of the exponential function, *b*, is called the **common ratio**. The common ratio will always be positive and will not equal one. The common ratio represents a growth or decay factor. Consider $d(x) = 10(2)^x$. For *d*(*x*), the common ratio is 2, and for every one-unit increase in *x*, *d*(*x*) doubles as shown in the table below.

x	$d(x)$
0	10
1	20
2	40
3	80

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If the common ratio is greater than one, the function will increase (maintaining our assumptions that *a* and *k* are positive). If the common ratio is less than one, the function will decrease. Consider $Q(x)=10(0.5)^x$ and the table below.

x	$Q(x)$
0	10
1	5
2	2.5
3	1.25

The common ratio can be written as 1 + *r* where *r* represents a **percent of change** written as a decimal. If the percent of change is a percent of decrease, *r* is negative. If the percent of change is a percent of increase, *r* is positive. In $d(x)=10(2)^x$, the function grows by 100% as *x* increases by one; thus, *r* = 1 and *b* = 2. In $Q(x)=10(0.5)^x$, the function decays by 50% as *x* increases by one; thus, *r* = –0.5 and *b* = 0.5.

Exponential functions change according to a *constant relative* rate. Consider $F(x)=1{,}000(1.4)^x$ and the table below.

x	$F(x)$
0	1,000
1	1,400
2	1,960
3	2,744

As the *x*-values increase by one, the function increases by different amounts. As *x* increased from zero to one, the function increased by 400 from 1,000 to 1,400. As *x* increased from one to two, the function increased, not by 400, but by 560 from 1,400 to 1,960. Obviously, the function is not increasing by a constant amount. If it did, it would be a linear function. The function does, however, increase by a constant *relative* amount. As *x* increased from zero to one, the function increased by 40%, increasing from 1,000 to 1,400. Note that 40% of 1,000 is 400, and 1,000 + 400 = 1,400. As *x* increased from one to two, the function increased again by 40%, increasing from 1,400 to 1,960. Note that 40% of 1,400 is 560, and 1,400 + 560 = 1,960.

For the function $P(x)=500(0.8)^x$, the initial amount is 500. Note that the common ratio is 0.8. Recall that *b* = 1 + *r*. Substituting 0.8 for *b* and solving for *r* finds the percent of change:

$$\begin{aligned} b &= 1+r \\ 0.8 &= 1+r \\ 0.8-1 &= r \\ -0.2 &= r \end{aligned}$$

Thus, the percent of change is –20%.

Instruction: *Introducing Exponential Functions*

Example 1
The Common Ratio

Consider $f(x)=10(0.8)^x$. What is the common ratio and the percent of change associated with *f*(*x*)?

The common ratio is the base of an exponential function. In a function of the form $y(x)=ab^{kx}$ where $a > 0$, $k > 0$, $b > 0$, and $b \neq 1$, the base *b* is the common ratio, and $b = 1 + r$ where *r* is the percent of change as a decimal and is negative if the percent of change is a percent of decrease but positive if the percent of change is a percent of increase. For *f*(*x*), the common ratio is 0.8 and the percent of change is a 20% decrease as shown below.

$$
\begin{aligned}
&b=1+r\\
&0.8=1+r\\
&0.8-1=r\\
&-0.2=r\\
&20\% \text{ decrease } = r
\end{aligned}
$$

Example 2
The Initial Amount

Consider $p(x)=6(2)^x$. What is the initial amount of *p*(*x*)? What is the *y*-intercept of *p*(*x*)?

The coefficient to the base is called the initial amount. In a function of the form $y(x)=ab^{kx}$, *a* is the initial amount. The initial amount corresponds to the *y*-intercept barring any reflection or transformation of $y(x)=ab^{kx}$ where $a > 0$, $k > 0$, $b > 0$, and $b \neq 1$. For *p*(*x*), the initial amount is 6. The *y*-intercept of *p*(*x*) is (0,6).

Example 3
The Asymptotic Value

Consider $q(x)=5(1.5)^x-2$. What is the asymptotic value of *q*(*x*)? What is the *y*-intercept?

In a function of the form $y(x)=ab^{kx}$ where $a > 0$, $k > 0$, $b > 0$, and $b \neq 1$, zero is the asymptotic value, meaning that the graph will approach zero as *x* approaches infinity if $b < 1$ or as *x* approaches negative infinity if $b > 1$. The asymptotic value changes with vertical translations so that the asymptotic value of $y(x)+c$ is *c*. For *q*(*x*), the asymptotic value is –2. The *y*-intercept of *q*(*x*) is (0,3).

Example 4
Growth versus Decay

Consider $y(x) = y_0 \cdot (1.2)^x$. Will the function increase or decrease?

A function of the form $f(x) = ab^{kx}$ where $a > 0$ and $k > 0$ will increase if $b > 1$. The function $y(x) = y_0 \cdot (1.2)^x$ will increase because $1.2 > 1$.

Example 5
Growth versus Decay

Consider $Q(x) = Q_0 \cdot 2^x$. Will $Q(-x)$ increase or decrease?

A function of the form $f(x) = ab^{kx}$ where $a > 0$ and $k > 0$ will increase if $b > 1$, but it will decrease if $b < 1$. The function $Q(x) = Q_0 \cdot 2^x$ increases because $2 > 1$. However, the reflection over the *y*-axis, denoted $Q(-x)$, will decrease.

Recall that $a^{-r} = \left(\frac{1}{a}\right)^r$ for all non-zero values of *a* and rewrite $Q(-x)$ so that the coefficient to *x* is positive.

$$Q(-x) = Q_0 \cdot 2^{-x} = Q_0 \cdot \left(\frac{1}{2}\right)^x$$

Since $1/2 < 1$, $Q(-x)$ decreases.

Practice Set 3.1

Fill in the blanks associated with each function.

1 $\sigma(x) = 450 \cdot 3^{x}$

A. The common ratio of $\sigma(x)$ is ____________________.
B. The percent of change of $\sigma(x)$ is _________________.
C. The asymptotic value of $\sigma(x)$ is __________________.
D. The initial amount of $\sigma(x)$ is ____________________.

2 $N(x) = 7.5(1.001)^{x}$

A. The common ratio of $N(x)$ is ____________________.
B. The percent of change of $N(x)$ is _________________.
C. The asymptotic value of $N(x)$ is __________________.
D. The initial amount of $N(x)$ is ____________________.

3 $v(x) = 100(.7)^{x} - 10$

A. The common ratio of $v(x)$ is ____________________.
B .The percent of change of $v(x)$ is ____________ (Hint: the percent of change for $v(x)$ will be negative).
C. The asymptotic value of $v(x)$ is __________________.
D. The initial amount of $v(x)$ is ____________________.

4 $T(x) = 88 \cdot (.92)^{x} + 12$

A. The common ratio of $T(x)$ is ____________________.
B. The percent of change of $T(x)$ is _________________.
C. The asymptotic value of $T(x)$ is __________________.
D. The initial amount of $T(x)$ is ____________________.

5 $Q(x) = (1.8)^{x}$

A. The common ratio of $Q(x)$ is ____________________.
B. The percent of change of $Q(x)$ is _________________.
C. The asymptotic value of $Q(x)$ is __________________.
D. The initial amount of $Q(x)$ is ____________________.

#1 A) 3, B) 200%, C) 0, D) 450
#2 A) 1.001, B) 0.1%, C) 0, D) 7.5
#3 A) 0.7, B) –30%, C) –10, D) 90
#4 A) 0.92, B) –8%, C) 12, D) 100
#5 A) 1.8, B) 80%, C) 0, D) 1

Study Exercise 3.1

Problems

Fill in the blanks associated with each function.

#1 $E(x) = 24 \cdot (1.1)^x$

The common ratio of $E(x)$ is ____________________.
The percent of change of $E(x)$ is _________________.
The y-intercept of $E(x)$ is ______________________.
The initial amount of $E(x)$ is ____________________.

#2 $Y(x) = 2(0.87)^x + 3$

The common ratio of $Y(x)$ is ____________________.
The percent of change of $Y(x)$ is _________________.
The y-intercept of $Y(x)$ is ______________________.
The asymptotic value of $Y(x)$ is ____________________.

Lecture 3.2

College Algebra

Instruction: ***Deriving Exponential Formulas from Story Problems***

If the exponential function is not given in the problems for this section, then assume that the exponential function will be of the form $y(x) = a \cdot b^x$ where a is a positive real number and b is a positive real number not equal to one.

Consider the description:

> A furnace heats a house to the constant temperature of 80^o Fahrenheit. The furnace breaks down at midnight. The weatherman asserts that the temperature outside will remain a constant 10^o Fahrenheit over the next several days. The building, of course, starts losing heat immediately after the furnace quits. The difference between the building's temperature and the outside temperature (70 at the moment the furnace quits) decreases by 10% every hour.

The above anecdote describes the declining temperature of the building. Since the ratio of temperatures from consecutive equally space readings is constant, an exponential function can be derived to represent the loss of heat. To derive the formula, one must identify 1) the independent variable (usually time) 2) the dependent variable 3) the initial amount & 4) the common ratio of decay or growth.

In the story of the heated house, the independent variable is time. Since the loss of heat is related to hours, it makes sense to let x represent the hours since the furnace quit. The dependent variable is the difference between the building's temperature and the temperature outside, which depends upon the amount of time that has elapsed since the furnace quit, so y represents the difference between the temperatures. The initial amount, therefore, is 70 because that is the difference between the ambient temperature and the temperature of the house when the house started losing heat. The common ratio of decay is .9 because the difference in temperatures drops every hour to 90% of the difference the previous hour. Thus, the formula for the difference between the temperatures is
$y = 70(.9)^x$, where x represents the time in hours since the furnace quit.

Consider the description:

> A fish and wildlife ranger stocks a pond with 500 fish. He then uses reliable statistical methods to count the population for the next four months with the following results:

month	fish population
1	525
2	551
3	579
4	608.

In the fish story, the independent variable, x, represents time in months. The dependent variable, y, represents the population of fish. Since the pond was initially stocked with 500 fish, the initial amount is 500. The common ratio is not stated, but it can be derived using the estimates of the fish population given in the table. Divide each population by the previous population. If the quotient remains constant, the growth is exponential, and the quotient will represent the common ratio:

608/579	1.05
579/551	1.0508
551/525	1.0495
525/500	1.05

Since the quotients hover at the 1.05, one can assume that the population grows exponentially with a common ratio of 1.05. Thus, the formula: $y = 500(1.05)^x$ where x represents the time in months since the pond was stocked.

Instruction: *Exponential Modeling*

Example 1
Deriving Exponential Models

> Skin injuries heal at an exponential rate. If a wound covers five square centimeters and the size of the wound decreases by 30% per day, write an exponential function that gives the size of the wound in terms of days since injury.

The function gives the size of the wound and has an initial value of five square centimeters. If the wound decreases by 30%, the common ratio will be 0.7 as shown below.

$$b = 1 + r$$
$$b = 1 + (-0.3)$$
$$b = 0.7$$

The function can be expressed as $W(x) = 5(0.7)^x$ where x represents days of healing and $W(x)$ represents the size of the wound in square centimeters.

Example 2
Deriving Exponential Models

> The population of a city according to a 1980 census was 20,000. By the 1990 census, the city had grown to 23,000, and by the 2,000 census, the city had grown to 26,450. Assume that the city will continue to grow at a constant relative rate. Write a function that gives the population in terms of time.

Choosing 1950 as a starting point in time, the population function will have an initial value of 20,000. The common ratio equals a ratio of values over even increments in time.

$$b = \frac{\text{value}}{\text{previous value}} \qquad b = \frac{\text{value}}{\text{previous value}}$$
$$b = \frac{23{,}000}{20{,}000} \qquad b = \frac{26{,}450}{23{,}000}$$
$$b = 1.15 \qquad b = 1.15$$

The function can be expressed as $P(x) = 20{,}000(1.15)^x$ where x represents decades since 1950 and $P(x)$ represents the population of the city. The function could also be expressed as $P(x) = 20{,}000(1.15)^{\frac{x}{10}}$ where x represents years (instead of decades) since 1950.

Practice Set 3.2

1 An appraisal company uses a function to asses the value in dollars of a property *x* years after its purchase for *a* dollars, assuming an annual rate of inflation, *r*. Write an exponential function that assesses the value of a house *x* years after its purchase for $110,000, assuming a 3% annual rate of inflation.

2 The population of a small town in 1994 was 15.5 thousand and growing at an annual rate of approximately 0.5% or 0.005. Write an exponential function that shows the town's population in thousands in terms of the number of years since 1994.

3 The World Health Organization estimated the number of people infected by a disease to be about 2 million in 2001. The World Health Organization further estimated that the number of people infected was growing by 1.7% annually. Write an exponential function that shows the growth in millions of people infected by the disease in terms of the number of years since 2001.

4 A farmer builds a rookery on two acres of land. The farmer puts 240 chickens on the land. A month later, he notes that he only has 228 chickens. Another month later, he notes that his chickens have been reduced to 217. Assume the approximate relative rate of decay continues and write an exponential function that gives the number of chickens in terms of the number of months since the farmer built his rookery.

5 According to a census taken in 1940, the population of a village was 400 people. According to a census in 1950, the population had increased to 440. In 1960, the population had increased again to 484. Finally, in year 1970, the population was measured at 532. Use 1940 as a starting point and derive an exponential function that depicts the population in the village in terms of the number of decades since 1940 assuming a constant relative rate of growth.

6 Use the information from problem five and write an exponential function that gives the population in terms of the number of years since the census was taken assuming a constant relative rate of growth.

7 Use the information from problem five and write an exponential function that gives the population in terms of the number of months since the census was taken assuming a constant relative rate of growth.

8 A researcher studies the use of computerized accounting software in homes. The original percentage of American households using the studied products at the beginning of the study was 2.8%. Reliable research shows that the percentage tends to increase by .4% every month. Write a function that predicts the percent of homes using the studied software products in terms of the months since the study began.

9 Scientists recorded a population of 2,000 frogs in a habitat before a chemical plant opened near the habitat. One month after the plant opened, the number of frogs had decreased to 1,900. Two months later, there were only 1,805 frogs. Write a function that gives the number of frogs in terms of the number of *days* since the chemical plant opened assuming a constant relative rate of decline.

<u>ANSWERS</u>

#1 $v(x) = 110{,}000(1.03)^x$ where *x* represents the years elapsed since purchase and *v*(*x*) represents the value of the house.

#2 $p(x) = 15.5(1.005)^x$ where *x* represents the years elapsed since 1994 and *p*(*x*) represents the population of the town.

#3 $I(x) = 2(1.017)^x$ where *x* represents the years elapsed since 2001 and *I*(*x*) represents the number of infected subjects.

#4 $r(x) = 240(.95)^x$ where *x* represents the months elapsed since stocking the rookery and *r*(*x*) represents the number of birds.

#5 $P(x) = 400(1.1)^x$ where *x* represents the decades elapsed since the census and *P*(*x*) represents the population of the village.

#6 $P(x) = 400(1.1)^{x/10}$ where *x* represents the years elapsed since the census and *P*(*x*) represents the population of the village.

#7 $P(x) = 400(1.1)^{\frac{x}{120}}$ where *x* represents months elapsed since the census and *P*(*x*) represents the population of the village.

#8 $H(x) = 2.8(1.004)^x$ where *x* represents months elapsed since the study began and *H*(*x*) represents the percent of homes using the studied products.

#9 $N(x) = 2{,}000(0.95)^{x/30}$ where *x* represents the days since the chemical plant opened and *N*(*x*) represents the number of frogs in the habitat.

Practice Set 3.2_Supplemental

1 A victim suffers an abrasion that measures 50 square centimeters. Each day 14% of the injured area heals. If 7 sq. cm. heals the first day, write an exponential function predicting the square area of the injury that heals in terms of days after day one.

2 A victim suffers an abrasion that measures 50 square centimeters. Each day 14% of the injured area heals. Write an exponential function predicting the size of the abrasion in terms of days since the injury.

3 The number of gadgets that Batman can affix to his utility belt grows by 10% with every visit to the Bat Cave. If Batman's utility belt can currently hold ten gadgets, write an exponential function that predicts the number of gadgets his utility belt can hold after *x* visits to the Bat Cave.

4 The WHO estimates the number of people infected by a disease to be about 100,000 in 2005. The WHO further estimates that the number of people infected is growing by 9% annually. Write an exponential function that shows the number of people infected by the disease in terms of the number of years since 2005.

5 A wildlife ranger stocks a pond with 1,000 bass. A month later, the ranger uses reliable statistical methods to determine that the population of bass has decreased to 875. Assume that the population is decreasing by a constant relative rate and write an exponential function that gives the population of bass in terms of months elapsed since the ranger stocked the pond.

6 According to a census taken in 1990, the population of a city was 180,000 people. According to a census in 2000, the population had increased to 212,400. In 2010, the population had increased again to 250,632. Use 1990 as a starting point and derive an exponential function that depicts the population of the city in terms of the number of decades since 1990 assuming a constant relative rate of growth.

7 Use the information from problem six and write an exponential function in terms of *years* since 1990.

8 The typical human body discharges 12% of happinol every hour. Write an exponential function that gives the amount of happinol in a patient *x*-hours after a nurse injects the patient with 500 mg.

9 Use the information from problem eight and write an exponential function in terms of *minutes* since injection.

10 Scientists recorded a population of 800 frogs in a pond and note that the number of frogs decreases by 50% every month. Write an exponential function that predicts the number of frogs in terms of years since the initial count.

ANSWERS

#1 $s(x) = 7(1.14)^x$ where *x* represents the days elapsed after day one of healing, and *s*(*x*) represents the healed area.

#2 $A(x) = 50(0.86)^x$ where *x* represents the days elapsed since the injury, and *A*(*x*) represents the size of the abrasion in square centimeters.

#3 $G(x) = 10(1.10)^x$ where *x* represents the number of visits to the Bat Cave, and *G*(*x*) represents the number of gadgets that will fit on the belt.

#4 $D(x) = 100{,}000(1.09)^x$ where *x* represents the years elapsed since 2005, and *D*(*x*) represents the number of people infected.

#5 $f(x) = 1{,}000(0.875)^x$ where *x* represents the months elapsed since stocking the pond, and *f*(*x*) represents the population of bass.

#6 $P(x) = 180{,}000(1.18)^x$ where *x* represents the decades elapsed since 1990, and *P*(*x*) represents the population of the city.

#7 $P(x) = 180{,}000(1.18)^{\frac{x}{10}}$ where *x* represents years elapsed since 1990, and *P*(*x*) represents the population of the village.

#8 $H(x) = 500(0.88)^x$ where *x* represents hours elapsed since injection, and *H*(*x*) represents the milligrams of happinol in the patient.

#9 $H(x) = 500(0.88)^{\frac{x}{60}}$ where *x* represents minutes elapsed since injection, and *H*(*x*) represents the milligrams of the drug in the patient.

#10 $F(x) = 800\left(\frac{1}{2}\right)^{12x}$ where *x* represents years elapsed, and *F*(*x*) represents the number of frogs.

Study Exercise 3.2

Problems

#1 According to a census taken in 1900, the population of a village was 500 people. According to a census in 1910, the population had increased to 600. In 1920, the population had increased again to 720. Use 1900 as a starting point and derive an exponential function that yields the population of the village in terms of decades elapsed since 1900 assuming that the relative rate of increase remains steady. For extra thought, predict the population in the year 2020.

#2 A beacon uses a battery for power supply. The wattage output of the battery decays by 5% every year. If the battery supplies 5,000 kilowatts of power when first placed into service, write a function that yields the battery's output during the service of the beacon in terms of years deployed. For extra thought, determine the approximate length of service of the beacon if it requires a minimum of 550 kilowatts to operate. HINT: A table of the values associated with the function will help.