Lecture 3.10

College Algebra

Instruction: ***Solving Exponential Equations Using Logarithms***

This lecture uses a four-step process to solve exponential equations:

1 Isolate the base if possible.
2 Take the natural log or common log of both sides.
3 Use properties of logarithms to remove variables from the exponent.
4 Solve for the unknown.

Consider the following exponential equation:

$$\frac{1}{8}(2)^{2x+1} - 3 = 1$$

Following the rubric above, the base should be isolated. In other words, algebraic transformations should be performed so that the expression 2^x stands alone on the left side of the equation:

$$\frac{1}{8}(2)^{2x+1} - 3 = 1$$
$$\frac{1}{8}(2)^{2x+1} = 1 + 3$$
$$\frac{1}{8}(2)^{2x+1} = 4$$
$$8 \cdot \left(\frac{1}{8}(2)^{2x+1} = 4\right)$$
$$(2)^{2x+1} = 32$$

Now that the base is isolated, the second step asserts that the natural log should be taken of both sides of the equation:

$$2^{2x+1} = 32$$
$$\ln 2^{2x+1} = \ln 32$$

The third step requires the use of logarithmic properties in order to remove the variable from the exponent. In this example, the property used is the property, $\log_b x^y = y \cdot \log_b x$:

$$\ln 2^{2x+1} = \ln 32$$
$$(2x+1)\ln 2 = \ln 32$$

Finally, simple algebra solves the resulting equation:

$$(2x+1)\ln 2 = \ln 32$$
$$\frac{(2x+1)\ln 2}{\ln 2} = \frac{\ln 32}{\ln 2}$$

Use a calculator to evaluate $\frac{\ln 32}{\ln 2}$, which equals 5.

$$2x + 1 = 5$$
$$2x = 4$$
$$x = 2$$

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For another example, consider the equation, $5^{2x-1} = 10^x$. In this example, the bases are isolated, so the equation is ready for step two. Since one of the bases is ten, taking the common log of both sides is recommended over taking the natural log of both sides:

$$5^{2x-1} = 10^x$$

$$\log 5^{2x-1} = \log 10^x$$

Now, the exponents can be removed from the expressions. On the left, the exponent can be removed using the property, $\log_b x^y = y \cdot \log_b x$:

$$\log 5^{2x-1} = \log 10^x$$

$$(2x-1)\log 5 = \log 10^x$$

On the right, the exponent can be removed using the property, $\log_b b^x = x$:

$$(2x-1)\log 5 = \log 10^x$$

$$(2x-1)\log 5 = x$$

The final step requires that the resulting equation be solved. Note that the log5 must be distributed across the binomial (2*x* – 1):

$$(2x-1)\log 5 = x$$

$$2x \cdot \log 5 - \log 5 = x$$ Distribute the log5 across 2x – 1.

$$2x \cdot \log 5 - x = \log 5$$ Subtract *x* from both sides and add log5 to both sides.

$$x(2\log 5 - 1) = \log 5$$ Factor out the greatest common factor, *x*, from the left side.

$$x = \frac{\log 5}{2\log 5 - 1}$$ ← Exact solution

$$x \approx \frac{.699}{2(.699) - 1}$$

$$x \approx 1.76$$ ← Approximate solution

Instruction: ***Solving Exponential Equations with Logarithms***

Example 1
Solving an Exponential Equation

Solve the equation for *x*: $9\left(2^{x+3}\right)+20=200$. Leave answer exact.

Isolate the base.

$$9\left(2^{x+3}\right)+20=200$$
$$9\left(2^{x+3}\right)=200-20$$
$$\frac{9\left(2^{x+3}\right)}{9}=\frac{180}{9}$$
$$2^{x+3}=20$$

Make both sides the argument of a logarithm.

$$\ln\left(2^{x+3}\right)=\ln(20)$$

Remove the exponent from the argument of the logarithm by applying the property of logarithms that states $\log_b\left(M^p\right)=p\cdot\log_b(M)$.

$$(x+3)\ln(2)=\ln(20).$$

Solve the linear equation.

$$(x+3)\ln(2)=\ln(20)$$
$$x+3=\frac{\ln 20}{\ln 2}$$
$$x=\frac{\ln 20}{\ln 2}-3$$

Example 2
Solving an Exponential Equation

Solve the equation for *x*: $7^{8x}+5=83$. Round answer to nearest hundredth.

Isolate the base.

$$7^{8x}+5=83$$
$$7^{8x}=78$$

Make both sides the argument of a logarithm.

$$\ln\left(7^{8x}\right)=\ln(78)$$

Remove the exponent from the argument of the logarithm by applying the property of logarithms that states $\log_b\left(M^p\right)=p\cdot\log_b(M)$.

$$8x\ln(7)=\ln(78).$$

Solve the linear equation.

$$8x\ln(7)=\ln(78)$$
$$8x=\frac{\ln(78)}{\ln(7)}$$
$$x=\frac{1}{8}\cdot\frac{\ln(78)}{\ln(7)}$$

Use a calculator to find an approximation.

$$x=\frac{1}{8}\cdot\frac{\ln(78)}{\ln(7)}\approx 0.28$$

Example 3
Solving an Exponential Equation

Solve the equation for x: $1-e^{\frac{x}{3}}=-54$. Round answer to nearest thousandth.

Isolate the base.

$$1-e^{\frac{x}{3}}=-54$$
$$1+54=e^{\frac{x}{3}}$$
$$55=e^{\frac{x}{3}}$$

Make both sides the argument of a logarithm.

$$\ln(55)=\ln\left(e^{\frac{x}{3}}\right)$$

Remove the exponent from the argument of the logarithm by applying the property of logarithms that states $\log_b\left(b^p\right)=p$.

$$\ln(55)=\frac{x}{3}.$$

Solve the linear equation.

$$\ln(55)=\frac{x}{3}$$
$$3\ln(55)=x$$

Use a calculator to find an approximation.

$$x=3\ln(55)\approx 12.022$$

Practice Set 3.10A

Remember the properties of logarithms in the boxes below.

$$\log_b a^p = p \cdot \log_b a$$

$$\log_b b^p = p$$

Solve the equations below. Approximate to the nearest thousandths when necessary.

#1 $5^x + 5 = 130$

#2 $2^x - 9 = 23$

#3 $120 \cdot 22^x = 120$

#4 $3^{x/2} - 3 = 6$

#5 $8 \cdot (1.5)^x = 72$

#6 $8 \cdot 2^x = 1$

#7 $(2.02)^{x/2} - 15 = 0$

#8 $2^{4x} = 6$

#9 $(17)^x - 3 = 0$

#10 $e^x = 42$

ANSWERS

#1) $x = 3$
#2) $x = 5$
#3) $x = 0$
#4) $x = 4$
#5) $x \approx 5.419$
#6) $x = -3$
#7) $x \approx 7.703$
#8) $x \approx 0.646$
#9) $x \approx 0.388$
#10) $x \approx 3.738$

Practice Set 3.10B

Remember the property of logarithms in the box below.

Since $\log_b b^p = p$, $\ln e^p = p$

Solve the equations below. Approximate to the nearest thousandths when necessary.

#1 $e^x + 5 = 130$

#2 $e^x - 9 = 23$

#3 $120 \cdot e^x = 90$

#4 $e^{x/2} + 3 = 15$

#5 $9 \cdot (e)^x = 36$

#6 $4 \cdot e^x + 5 = 7$

#7 $(e)^{12x} = 4$

#8 $e^{\frac{x}{2,000}} = 2$

#9 $4e^x + 2 = 6$

#10 $e^{2x} = e^{22}$

ANSWERS

#1) $x \approx 4.828$
#2) $x \approx 3.466$
#3) $x \approx -0.288$
#4) $x \approx 4.97$
#5) $x \approx 1.386$
#6) $x \approx -0.693$
#7) $x \approx 0.116$
#8) $x \approx 1,386.294$
#9) $x = 0$
#10) $x = 11$

Practice Set 3.10C

Solve the story problems below. Approximate to reasonable quantities.

#1 The radioactive element polonium decays according to the law $Q(t) = Q_0 \cdot 2^{-\left(\frac{t}{140}\right)}$ where t represents days elapsed. If the amount of polonium left after 420 days is 65 mg, what was the initial amount present?

#2 Carbon-14 decays according to the function $Q(t) \approx Q_0 e^{-0.00012t}$ where t represents years elapsed. A skull from an archeological site has one-fourth the amount of C-14 that it originally contained. Determine the approximate age of the skull.

#3 Halley's law states that the barometric pressure (in inches of mercury) at an altitude of x miles above sea level is approximately given by the equation $p(x) = 29.92e^{-0.2x}$ where $x > 0$. If the barometric pressure as measured by a hot-air-balloonist is 18 inches of mercury, what is the balloonist's approximate altitude?

#4 At a restaurant, the chef determines that the soup should be served to customers at a temperature of no less than 160° F. It has been determined that the cooling rate for this soup is 0.21° F per minute. According to Newton's Law of Cooling, the temperature T of the soup t minutes after it is removed from a boiling pot is given by the model $T(t) = T_A + (T_0 - T_A)e^{-kt}$, $t > 0$, where T_A is the ambient temperature, k is the cooling rate, and T_0 is the initial temperature of the soup. After removing the soup from a boiling temperature of 212° F, approximately how many minutes may the soup stand in the 78° F kitchen before it must be served to the customer?

#5 The function $P(x) = P_0(1.5)^x$ represents the population growth of a species of bacteria where x represents hours elapsed since 8:00 am when the population was cultured and measured P_0. Determine the time of day when the bacteria population will double.

#6 Special ingredient-*X*, used to create the Power Puff Girls, decays according to the function $Q(t) = Q_0 2^{-0.1t}$ where t represents years elapsed. If the Professor infused Blossom with 12 milligrams of special ingredient-*X*, how many years of crime fighting will she enjoy before her special ingredient-*X* levels deplete below 2 milligrams?

ANSWERS

#1 The initial amount of polonium equals 520 milligrams.
#2 The skull is approximately 11,552 years old.
#3 The balloon has an altitude of about 2.5 miles.
#4 The soup should stand a maximum of 2 minutes and 20 seconds.
#5 The population will double at 9:42 am.
#6 Blossom will enjoy 25 full years of crime fighting before her special ingredient-*X* levels deplete to 2 milligrams.

Study Exercise 3.10

Problems

Solve the equations for the unknown. Round the answers to the nearest tenth when necessary.

#1 $\left(\frac{1}{3}\right)^x + 2 = 83$

#2 $e^{\frac{x}{2}} = 5$

#3 $4^{x+3} = 7^x$

#4 $100(1.08)^{4x} = 200$