

**Instruction: Rational Functions**

A rational function is a ratio of two polynomial functions. If  $P(x)$  and  $Q(x)$  are polynomial functions and  $Q(x)$  is not the zero polynomial, then  $R(x) = \frac{P(x)}{Q(x)}$  is a rational function defined for all values of  $x$  where  $Q(x) \neq 0$ .

**Instruction: Points of Discontinuity on the Graphs of Rational Functions**

If  $R(x) = \frac{P(x)}{Q(x)}$ , then the roots of  $Q(x)$  represent restrictions on the domain of  $R(x)$ . The graph of rational function will demonstrate one of two types of behavior at a domain restriction: a point of discontinuity or a vertical linear asymptote. Vertical asymptotes will be discussed below. A point of discontinuity is a point  $(x_a, y_a)$  where  $x_a$  is a root of  $Q(x)$ , that is,  $Q(x_a) = 0$ . The value  $y_a$  can be found by simplifying  $R(x)$  then evaluating the result at  $x_a$ . Consider  $f(x)$  below.

$$f(x) = \frac{x^2 + 2x + 1}{x + 1}$$

Note that  $x = -1$  is a restriction on the domain of  $f(x)$  because  $x = -1$  is a root of the function in the denominator. The function  $f(x)$  simplifies as below.

$$f(x) = \frac{x^2 + 2x + 1}{x + 1} = \frac{(x+1)\cancel{(x+1)}}{\cancel{x+1}}$$

$$f(x) = x + 1, x \neq -1$$

Evaluating the simplified function at the restriction yields the following.

$$f(x) = x + 1$$

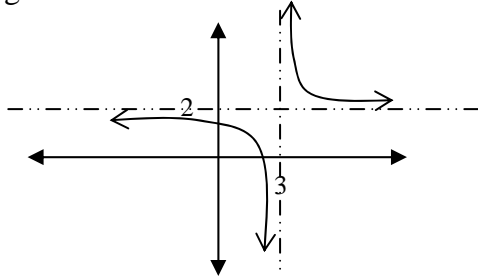
$$f(-1) = -1 + 1$$

$$f(-1) = 0$$

Thus, the point  $(-1, 0)$  is a point of discontinuity on the graph of  $f(x)$ .

**Instruction: Vertical Asymptotes of Rational Functions**

When a graph approaches a line, that line is a linear asymptote. The function below has a vertical asymptote of  $x = 3$ , and a horizontal asymptote at  $y = 2$ . Since asymptotes are not part of the graph, they are indicated using dotted lines.



Many rational functions have asymptotes. Finding the asymptotes is an important step in graphing a rational function, and this lecture will discuss how to find the vertical asymptotes of a rational function.

$$r(x) = \frac{x+1}{x^2-1}$$

Consider  $r(x)$  above. The vertical asymptotes of a rational function correspond to the roots of the denominator *after* the rational expression has been reduced. Consequently, the first step for finding the vertical asymptote of a rational function requires that the rational expression be reduced:

$$r(x) = \frac{x+1}{(x+1)(x-1)}$$

$$r(x) = \frac{\cancel{x+1}}{(\cancel{x+1})(x-1)}$$

$$r(x) = \frac{1}{(x-1)}, x \neq -1$$

Factoring the denominator shows that the rational expression reduces because the common factor  $(x+1)$  cancels. Now that the rational expression is reduced, solving for the roots of the denominator finds the vertical asymptotes:

$$x-1=0$$

$$x=1$$

Since the root of the denominator is 1, the function has a vertical asymptote at  $x = 1$ . The other domain restriction,  $-1$ , corresponds, not to a vertical asymptote but to a point of discontinuity, or a "hole" in the graph discussed in Section 2.14.

Consider  $t(x)$  below.

$$t(x) = \frac{x+5}{x^2-x-12}$$

Since  $t(x)$  does not reduce, finding the roots of  $x^2 - x - 12$  yields the vertical asymptotes:

Lecture 2.10

$$x^2 - x - 12 = 0$$

$$(x+3)(x-4) = 0$$

$$x+3=0 \quad \text{or} \quad x-4=0$$

$$x=-3 \quad \text{or} \quad x=4$$

So,  $t(x)$  has two vertical asymptotes:  $x = -3$  and  $x = 4$ .

**Instruction: Vertical Asymptotes**

**Example 1  
Finding Vertical Asymptotes**

In terms of the Cartesian plane, what describes a linear asymptote, a number or an equation?

A linear asymptote is a line, so it takes an equation, not simply a number, to describe an asymptote in terms of the Cartesian plane. Vertical asymptotes will take the form  $x = a$  where  $a$  is a real number non-reducible restriction on the domain of a rational function.

**Example 2  
Identifying Graphical Representations of Domain Restrictions**

Consider  $q(x) = \frac{x^2 + x - 12}{x^2 + 7x + 12}$ . Identify the vertical asymptotes and points of discontinuity of  $q(x)$ .

Reduce the rational function.

$$q(x) = \frac{x^2 + x - 12}{x^2 + 7x + 12} = \frac{\cancel{(x+4)}(x-3)}{\cancel{(x+4)}(x+3)} = \frac{x-3}{x+3}, x \neq -4$$

Set the denominator in the reduced form equal to zero and solve.

$$x + 3 = 0$$

$$x = -3$$

The equation  $x = -3$  describes the vertical asymptote of  $q(x)$ .

Next, evaluate the reduced function at the root of the eliminated factor. In other words, evaluate

$\frac{x-3}{x+3}$  at  $x = -4$  as below.

$$\frac{-4-3}{-4+3} = \frac{-7}{-1} = 7$$

The point  $(-4, 7)$  is a point of discontinuity on the graph of  $q(x)$ .

**Example 3**  
**Finding Vertical Asymptotes**

Consider  $h(x) = \frac{x+9}{2x^3 + 15x^2 - 29x - 18}$ . Identify the vertical asymptotes of  $h(x)$ .

To factor the denominator, divide synthetically using the denominator as the dividend and its possible rational roots as divisors. Divide until a zero appears in the last entry of the quotient row.

$$\begin{array}{r|rrrr}
 2 & 2 & 15 & -29 & -18 \\
 & & 4 & 38 & 18 \\
 \hline
 & 2 & 19 & 9 & 0
 \end{array}$$

The zero indicates that the divisor, 2, is a root, which by the Factor Theorem means  $(x-2)$  is a factor. Interpret the quotient row as a quadratic factor of the polynomial:  $2x^2 + 19x + 9$ . Factor the quadratic and reduce  $h(x)$ .

$$h(x) = \frac{\cancel{x+9}}{(x-2)(2x+1)\cancel{(x+9)}} = \frac{1}{(x-2)(2x+1)}$$

Set the denominator in the reduced form equal to zero and solve.

$$\begin{aligned}
 (2x+1)(x-2) &= 0 \\
 2x+1=0 & \quad x-2=0 \\
 2x &= -1 & \quad x &= 2 \\
 x &= -\frac{1}{2}
 \end{aligned}$$

The equations  $x = -\frac{1}{2}$  and  $x = 2$  describe the vertical asymptotes of  $h(x)$ .

**Example 4**  
**Finding Vertical Asymptotes**

Consider  $R(x) = \frac{5}{x^2 + 5}$ . Show that  $R(x)$  has no vertical asymptotes.

Vertical asymptotes take the form  $x = a$  where  $a$  is a real number non-reducible root of the denominator. Show that  $R(x)$  has no real roots.

$$x^2 + 5 = 0$$

$$x^2 = -5$$

$$\sqrt{x^2} = \pm\sqrt{-5}$$

$$x = \pm\sqrt{5 \cdot -1}$$

$$x = \pm\sqrt{5} \cdot i$$

Practice Set 2.10

Find the vertical asymptotes and points of discontinuity of the following rational functions.

#1  $f(x) = \frac{x+3}{x^2+11x+28}$

#2  $r(x) = \frac{x+2}{x^2+4x+3}$

#3  $R(x) = \frac{x^2-4}{6x^2-7x-5}$

#4  $q(x) = \frac{8}{x^2-4}$

#5  $Q(x) = \frac{x+2}{x^2-4}$

#6  $d(x) = \frac{x^2+2}{x^3+2x^2-5x-6}$  HINT: Use synthetic division to factor the denominator.

#7  $D(x) = \frac{x^3+2x^2-7x-2}{x^2+2}$

#8  $g(x) = \frac{x^3-8}{x-1}$

#9  $h(x) = \frac{12}{x^3+4x^2-x-4}$

#10  $p(x) = \frac{x^2-1}{x-1}$

ANSWERS

#1 VA:  $x = -4, x = -7$

#2 VA:  $x = -3, x = -1$

#3 VA:  $x = -\frac{1}{2}, x = \frac{5}{3}$

#4 VA:  $x = 2, x = -2$

#5 VA:  $x = 2$ , Point of Discontinuity:  $(-2, -\frac{1}{4})$

#6 VA:  $x = -3, x = -1, x = 2$

#7 The rational function does not have any domain restrictions.

#8 VA:  $x = 1$

#9 VA:  $x = -1, x = 1, x = -4$

#10 Point of Discontinuity:  $(1, 2)$

Assignment 2.10

**Problems**

Find the vertical asymptotes of the following rational functions.

1  $f(x) = \frac{x}{x-9}$

2  $g(x) = \frac{x+3}{x^2-x-30}$

3  $h(x) = \frac{x^2-4}{x^2+9x+14}$

4  $j(k) = \frac{k+2}{k^3+2k^2-4k-8}$

**NOTES**

Study Exercise 2.10

**Problems**

Find all the vertical asymptotes of the following rational functions. Be sure to use equations of the form  $x = a$  to describe a vertical asymptote.

#1  $f(x) = \frac{x}{2+x}$

#2  $R(x) = \frac{x+3}{x^2+x-6}$

#3  $Q(x) = \frac{x^2-4}{x^2+8x+12}$

#4  $y(x) = \frac{x-1}{x^3+2x^2-5x-6}$