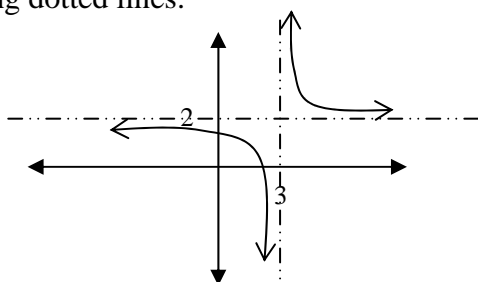


Instruction: Rational Functions

A rational function is a ratio of two polynomial functions. If $P(x)$ and $Q(x)$ are polynomial functions and $Q(x)$ is not the zero polynomial, then $R(x) = \frac{P(x)}{Q(x)}$ is a rational function defined for all values of x where $Q(x) \neq 0$.

Instruction: Vertical Asymptotes of Rational Functions

When a graph approaches a line, that line is an asymptote. The function below has a vertical asymptote of $x = 3$, and a horizontal asymptote at $y = 2$. Since asymptotes are not part of the graph, they are indicated using dotted lines.



Many rational functions have asymptotes. Finding the asymptotes is an important step in graphing a rational function, and this lecture will discuss how to find the vertical asymptotes of a rational function.

$$r(x) = \frac{x+1}{x^2-1}$$

Consider $r(x)$ above. The vertical asymptotes of a rational function correspond to the roots of the denominator *after* the rational expression has been reduced. Consequently, the first step for finding the vertical asymptote of a rational function requires that the rational expression be reduced:

$$r(x) = \frac{x+1}{(x+1)(x-1)}$$

$$r(x) = \frac{\cancel{x+1}}{(\cancel{x+1})(x-1)}$$

$$r(x) = \frac{1}{(x-1)}, x \neq -1$$

Factoring the denominator shows that the rational expression reduces because the common factor $(x+1)$ cancels. Now that the rational expression is reduced, solving for the roots of the denominator finds the vertical asymptotes:

$$x-1=0$$

$$x=1$$

Lecture 2.10

Since the root of the denominator is 1, the function has a vertical asymptote at $x = 1$. The other domain restriction, -1 , corresponds, not to a vertical asymptote but to a point of discontinuity, or a "hole" in the graph discussed in Section 2.14.

Consider $t(x)$ below.

$$t(x) = \frac{x+5}{x^2 - x - 12}$$

Since $t(x)$ does not reduce, finding the roots of $x^2 - x - 12$ yields the vertical asymptotes:

$$x^2 - x - 12 = 0$$

$$(x+3)(x-4) = 0$$

$$x+3 = 0 \quad \text{or} \quad x-4 = 0$$

$$x = -3 \quad \text{or} \quad x = 4$$

So, $t(x)$ has two vertical asymptotes: $x = -3$ and $x = 4$.

Instruction: Vertical Asymptotes

Example 1
Finding Vertical Asymptotes

In terms of the Cartesian plane, what describes a linear asymptote, a number or an equation?

A linear asymptote is a line, so it takes an equation, not simply a number, to describe an asymptote in terms of the Cartesian plane. Vertical asymptotes will take the form $x = a$ where a is a real number non-reducible restriction on the domain of a rational function.

Example 2
Finding Vertical Asymptotes

Consider $q(x) = \frac{x^2 + x - 12}{x^2 + 7x + 12}$. Identify the vertical asymptotes of $q(x)$.

Reduce the rational function.

$$q(x) = \frac{x^2 + x - 12}{x^2 + 7x + 12} = \frac{\cancel{(x+4)}(x-3)}{\cancel{(x+4)}(x+3)} = \frac{x-3}{x+3}, x \neq -4$$

Set the denominator in the reduced form equal to zero and solve.

$$x + 3 = 0$$

$$x = -3$$

The equation $x = -3$ describes the vertical asymptote of $q(x)$.

Example 3
Finding Vertical Asymptotes

Consider $h(x) = \frac{x+9}{2x^3 + 15x^2 - 29x - 18}$. Identify the vertical asymptotes of $h(x)$.

To factor the denominator, divide synthetically using the denominator as the dividend and its possible rational roots as divisors. Divide until a zero appears in the last entry of the quotient row.

$$\begin{array}{r|rrrr}
 2 & 2 & 15 & -29 & -18 \\
 & & 4 & 38 & 18 \\
 \hline
 & 2 & 19 & 9 & 0
 \end{array}$$

The zero indicates that the divisor, 2, is a root, which by the Factor Theorem means $(x - 2)$ is a factor. Interpret the quotient row as a quadratic factor of the polynomial: $2x^2 + 19x + 9$. Factor the quadratic and reduce $h(x)$.

$$h(x) = \frac{\cancel{x+9}}{(x-2)(2x+1)\cancel{(x+9)}} = \frac{1}{(x-2)(2x+1)}$$

Set the denominator in the reduced form equal to zero and solve.

$$\begin{aligned}
 (2x+1)(x-2) &= 0 \\
 2x+1=0 &\quad x-2=0 \\
 2x &= -1 &\quad x &= 2 \\
 x &= -\frac{1}{2}
 \end{aligned}$$

The equations $x = -\frac{1}{2}$ and $x = 2$ describe the vertical asymptotes of $h(x)$.

Example 4
Finding Vertical Asymptotes

Consider $R(x) = \frac{5}{x^2 + 5}$. Show that $R(x)$ has no vertical asymptotes.

Vertical asymptotes take the form $x = a$ where a is a real number non-reducible root of the denominator. Show that $R(x)$ has no real roots.

$$x^2 + 5 = 0$$

$$x^2 = -5$$

$$\sqrt{x^2} = \pm\sqrt{-5}$$

$$x = \pm\sqrt{5 \cdot -1}$$

$$x = \pm\sqrt{5} \cdot i$$

Practice Set 2.10

Find the vertical asymptotes of the following rational functions.

#1 $f(x) = \frac{x+3}{x^2+11x+28}$

#2 $r(x) = \frac{x+2}{x^2+4x+3}$

#3 $R(x) = \frac{x^2-4}{6x^2-7x-5}$

#4 $q(x) = \frac{8}{x^2-4}$

#5 $Q(x) = \frac{x+2}{x^2-4}$

#6 $d(x) = \frac{x^2+2}{x^3+2x^2-5x-6}$ HINT: Use synthetic division to factor the denominator.

#7 $D(x) = \frac{x^3+2x^2-7x-2}{x^2+2}$

#8 $g(x) = \frac{x^3-8}{x-1}$

#9 $h(x) = \frac{12}{x^3+4x^2-x-4}$

#10 $p(x) = \frac{x^2-1}{x-1}$

ANSWERS

#1 VA: $x = -4, x = -7$

#2 VA: $x = -3, x = -1$

#3 VA: $x = -\frac{1}{2}, x = \frac{5}{3}$

#4 VA: $x = 2, x = -2$

#5 VA: $x = 2$

#6 VA: $x = -3, x = -1, x = 2$

#7 VA: None, the rational function does not have any vertical asymptotes

#8 VA: $x = 1$

#9 VA: $x = -1, x = 1, x = -4$

#10 VA: None, the rational function does not have any vertical asymptotes

Study Exercise 2.10

Problems

Find all the vertical asymptotes of the following rational functions. Be sure to use equations of the form $x = a$ to describe a vertical asymptote.

#1 $f(x) = \frac{x}{2+x}$

#2 $R(x) = \frac{x+3}{x^2+x-6}$

#3 $Q(x) = \frac{x^2-4}{x^2+8x+12}$

#4 $y(x) = \frac{x-1}{x^3+2x^2-5x-6}$