Lecture 1.7

College Algebra

Instruction: *Inverse Functions*

Any one-to-one function has an inverse. A function is one-to-one if no two ordered pairs in the function have the same second component. In other words, one-to-one functions do not have any repeated values. Consequently, $y = x^2$ is not one-to-one nor does it have an inverse (unless its domain is restricted). A quick test for one-to-one correspondence is the horizontal-line test. If all the horizontal lines of the Cartesian plane (the *x*-*y* plane) intersect the function only once, the function is one-to-one and has an inverse.

If a function is a one-to-one function, then its inverse is the function formed by reversing all the ordered pairs. In other words, for the function defined by the ordered pairs (-2,6) and (4,8), the inverse has the ordered pairs (6,-2) and (8,4). If a function is labeled *f*, then its inverse is denoted f^{-1}.

Finding a function's inverse involves a four-step algebraic process. The first step is to make sure the function is one-to-one. If it is, rewrite it using y in place of the normal functional notation.

$$w(x) = \sqrt[3]{x+3}$$
$$y = \sqrt[3]{x+3}$$

Since reversing the ordered pairs forms the inverse, the second step requires the interchange of the *x* and *y* in the equation.

$$y = \sqrt[3]{x+3}$$
$$x = \sqrt[3]{y+3}$$

The third step requires solving the equation for *y*.

$$x = \sqrt[3]{y+3}$$
$$x^3 = \left(\sqrt[3]{y+3}\right)^3$$
$$x^3 = y+3$$
$$x^3 - 3 = y$$
$$y = x^3 - 3$$

Finally, replacing y with the inverse notation completes the process of finding the inverse.

$$y = x^3 - 3$$
$$w^{-1}(x) = x^3 - 3$$

Since w^{-1} represents the function formed by reversing the ordered pairs of *w*, the composition function $w(w^{-1}(x))$ should equal *x*.

Instruction: *Inverse Functions*

Example 1
Finding an Inverse Function

Given $f(x) = 5x - 7$, what is the inverse of $f(x)$?

To find the inverse, 1) rewrite the function dropping functional notation, 2) interchange *x* and *y*, 3) solve for *y*, and 4) rewrite using inverse function notation.

1) Drop function notation.

$$f(x) = 5x - 7$$
$$y = 5x - 7$$

2) Interchange *x* and *y*.

$$y = 5x - 7$$
$$x = 5y - 7$$

3) Solve for *y*.

$$x = 5y - 7$$
$$x + 7 = 5y$$
$$\frac{1}{5}(x+7) = \frac{1}{\not{5}}(\not{5}y)$$
$$\frac{1}{5}x + \frac{7}{5} = y$$

4) Rewrite using inverse function notation.

$$y = \frac{1}{5}x + \frac{7}{5}$$
$$f^{-1}(x) = \frac{1}{5}x + \frac{7}{5}$$

Example 2
Finding an Inverse Function

Given $g(x) = -\frac{2}{5}x + 1$, show that $g^{-1}(x)$ decreases along some interval.

Either find $g^{-1}(x)$ or generate a table of values for $g(x)$ and interchange the *x* and *y*-values to see a table of values for $g^{-1}(x)$.

Generate a table of values for $g(x)$.

x	$-\frac{2}{5}x+1$	$g(x)$
0	$-\frac{2}{5}(0)+1$	1
1	$-\frac{2}{5}(1)+1$	0.6
2	$-\frac{2}{5}(2)+1$	0.2
3	$-\frac{2}{5}(3)+1$	–0.2

2) Interchange *x* and *y*.

x	$g^{-1}(x)$
1	0
0.6	1
0.2	2
–0.2	3

3) Re-order the table so that the *x*-values increase and note that as the *x*-values increase, $g^{-1}(x)$ decreases.

x	$g^{-1}(x)$
–0.2	3
0.2	2
0.6	1
1	0

Example 3
Finding Range Restrictions Using the Inverse Function

Given $r(x)=\dfrac{2x}{7-5x}$. What restrictions are there--if any--on the range of $r(x)$?

The domain restriction on $r(x)$ is readily identifiable by examining the denominator. The x-values that render the denominator zero are domain restrictions because division by zero is undefined, so $7/5$ is a domain restriction. The question, however, asks for range restrictions. Section 2.11 deals with some rules that will make the range restrictions just as apparent as domain restrictions. Without these rules on hand, remember that the range of a function equals the domain of its inverse. Finding the domain restrictions on r^{-1} finds the range restrictions on r. Start by interchanging y and x.

$$r(x)=\frac{2x}{7-5x}$$
$$y=\frac{2x}{7-5x}$$
$$x=\frac{2y}{7-5y}$$

To solve for y, multiply both sides by the denominator, then move all the y-variables to one side and factor out a y as shown below.

$$(7-5y)\cdot x=\frac{2y}{\cancel{7-5y}}\cdot\cancel{(7-5y)}$$
$$7x-5yx=2y$$
$$7x=2y+5yx$$
$$7x=y(2+5x)$$
$$\frac{7x}{(2+5x)}=\frac{y\cancel{(2+5x)}}{\cancel{(2+5x)}}$$
$$\frac{7x}{(2+5x)}=y$$
$$\therefore$$
$$r^{-1}(x)=\frac{7x}{(2+5x)}$$

The domain restriction of r^{-1} is $-2/5$, so the range restriction of r is $-2/5$. In set interval notation, the range of r is: $R=(-\infty,-2/5)\cup(-2/5,\infty)$.

Example 4
Verifying Inverses

Given $p(x) = \sqrt[3]{x+3}$ and $q(x) = x^3 - 3$, show that $p(x)$ and $q(x)$ are inverse functions.

If $q(x)$ is the inverse of $p(x)$, then the range values of q equal the domain values of p; therefore, substituting the y-values of q into p for the x-values yields the x-values of q. Accordingly, $p \circ q(x) = x$.

$$p \circ q(x) = \sqrt[3]{\left(x^3 - 3\right) + 3}$$
$$p \circ q(x) = \sqrt[3]{x^3 - 3 + 3}$$
$$p \circ q(x) = \sqrt[3]{x^3}$$
$$p \circ q(x) = x$$

Practice Set 1.7

#1 Given $k(x) = \sqrt{x+5}$, find $k^{-1}(x)$.

#2 Given $j(x) = 6x$, find $j^{-1}(x)$.

#3 Given $m(x) = x^2 - 1$, $x \geq 0$; find $m^{-1}(x)$.

#4 Given $P(x) = \frac{5}{2}x^2$, $x \geq 0$; find $P^{-1}(x)$.

#5 Given $q(x) = \frac{x}{x+1}$, find $q^{-1}(x)$.

#6 Given $n(x) = 5 - x$, evaluate $n^{-1}(-3)$.

#7 Given $p(x) = x^3$, evaluate $p^{-1}(-8)$.

#8 Given $f(x) = 2x$ and $g(x) = \frac{1}{2}x$, evaluate $(f \circ g)(x)$ to determine if *f* and *g* are inverses.

#1 $k^{-1}(x) = x^2 - 5$, #2 $j^{-1}(x) = \frac{1}{6}x$, #3 $m^{-1}(x) = \sqrt{x+1}$, #4 $P^{-1}(x) = \sqrt{\frac{2}{5}x}$,

#5 $q^{-1}(x) = \frac{-x}{x-1}$, #6 $n^{-1}(-3) = 8$, #7 $p^{-1}(-8) = -2$,

#8 $(f \circ g)(x) = 2\left(\frac{1}{2}x\right) = x$

$\therefore g(x) = f^{-1}(x)$

Study Exercise 1.7

Problems

#1 Find $f^{-1}(x)$, given $f(x)=\sqrt[3]{x+1}$.

#2 Use a table or graph of $g^{-1}(x)$ to show that $g^{-1}(x)$ is an increasing function given $g(x)=\frac{1}{2}x-1$.