Lecture 1.6

College Algebra

Instruction: *Operations with Functions*

$m(x) = 2x + 3 \qquad n(x) = x^2 + 2x + 1$

Functions can undergo the four basic operations. For example, $m(x)$ and $n(x)$ can be added. Symbolically, $m(x) + n(x)$ or $(m + n)(x)$ denotes adding the functions:

$$(m+n)(x) = 2x + 3 + x^2 + 2x + 1$$
$$(m+n)(x) = x^2 + 4x + 4.$$

Functions can also be multiplied, thus:

$$(mn)(x) = (2x+3)(x^2 + 2x + 1)$$
$$(mn)(x) = 2x^3 + 4x^2 + 2x + 3x^2 + 6x + 3$$
$$(mn)(x) = 2x^3 + 7x^2 + 8x + 3.$$

Subtraction is performed similarly:

$$(m-n)(x) = 2x + 3 - \left(x^2 + 2x + 1\right)$$
$$(m-n)(x) = 2x + 3 - x^2 - 2x - 1$$
$$(m-n)(x) = -x^2 + 2.$$

Likewise, functions can be divided:

$$(m/n)(x) = \frac{2x+3}{x^2+2x+1}, \quad x \neq -1$$

Multiplication and addition with functions is commutative, so $(mn)(x) = (nm)(x)$ and $(m + n)(x) = (n + m)(x)$. Subtraction and division are not commutative, so $(m/n)(x) \neq (n/m)(x)$ and $(m - n)(x) \neq (n - m)(x)$.

Instruction: *Composition Functions*

$m(x) = 2x + 3 \qquad n(x) = x^2 + 2x + 1$

One function can be composed with another. For instance, m can be composed with n. Composition can be denoted using functional notation: $m(n(x))$. Alternatively, composition can be denoted using the symbol: $(m \circ n)(x)$, so $m(n(x)) = (m \circ n)(x)$ and $n(m(x)) = (n \circ m)(x)$. Composition requires one function to be used as the input value into the other function:

$$m(n(x)) = 2\left(x^2 + 2x + 1\right) + 3$$
$$m(n(x)) = 2x^2 + 4x + 2 + 3$$
$$m(n(x)) = 2x^2 + 4x + 5.$$

Alternatively:

$$(n \circ m)(x) = (2x+3)^2 + 2(2x+3) + 1$$
$$(n \circ m)(x) = (2x+3)(2x+3) + 4x + 6 + 1$$
$$(n \circ m)(x) = 4x^2 + 6x + 6x + 9 + 4x + 7$$
$$(n \circ m)(x) = 4x^2 + 16x + 16.$$

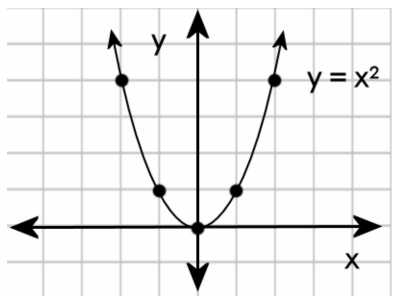
A function can be composed with itself:

$$(m \circ m)(x) = 2(2x+3) + 3$$
$$(m \circ m)(x) = 4x + 6 + 3$$
$$(m \circ m)(x) = 4x + 9.$$

Instruction: *Transformation*

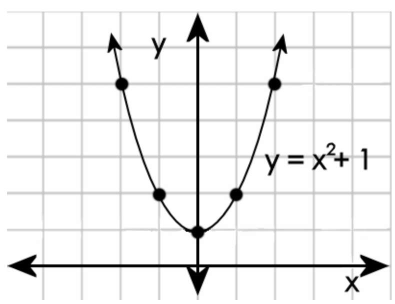
Consider the function $q(x) = x^2$ whose table and graph are given below.

x	$q(x)$
–2	4
–1	1
0	0
1	1
2	4



Transformations of functions involve manipulations of some basic function like $q(x)$. There are three types of transformations. First, the function can be shifted either vertically, horizontally, or both. Vertical and horizontal shifts are called translations. A vertical shift occurs when some number, c, is added to $q(x)$. Vertical shifts can be denoted as $q(x) + c$. Or, if $q(x) = x^2$, then a vertical shift of $q(x)$ could be indicated as $s(x) = x^2 + c$ (i.e., $q(x) + c = s(x)$). If c is positive, the shift will be up. If c is negative, the shift will be down. For example, $q(x) + 1$ is graphed below.

x	$q(x) + 1$
–2	5
–1	2
0	1
1	2
2	5

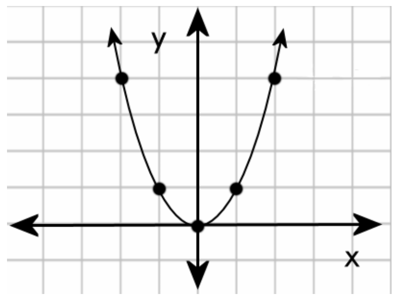


Horizontal shifts occur when some number, c, is added to the independent variable (x). Horizontal shifts can be denoted as $q(x + c)$. Or, if $q(x) = x^2$, then a horizontal shift could be indicated as $s(x) = (x + c)^2$. If c is positive, the shift will be to the left (not right as one might think). If c is negative, the shift will be to the right (not to the left as one might think).
For example, $q(x + 1)$:

x	$q(x+1)$
-2	1
-1	0
0	1
1	4
2	9

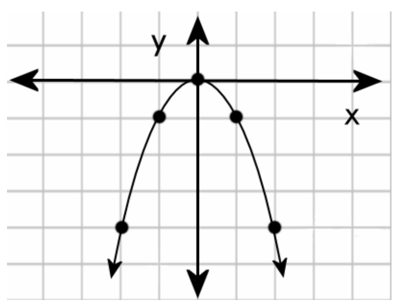
Reflections represent a second type of transformation. Reflection occurs when either all the x-values are multiplied by a negative one or all the y-values are multiplied by a negative one. The former type, reflects the graph across the y-axis. In layman's terms, reflection across the y-axis left-right reflection because all the points of the graph reflect to the left or to the right. Reflection across the y-axis can be denoted as $q(-x)$. Sometimes $f(x) = f(-x)$. In these cases, the functions are called even functions. Our $q(x)$ is an even function because $x^2 = \left(-x^2\right)$.

x	$q(x)$	$q(-x)$
–2	4	4
–1	1	1
0	0	0
1	1	1
2	4	4



Reflection across the x-axis (up-down reflection) can be denoted as $-q(x)$. Or, if $q(x) = x^2$ and $s(x) = -x^2$, then $s(x)$ is a reflection of $q(x)$ over the x-axis. Sometimes $f(-x) = -f(x)$. In these cases, the functions are called odd functions. The basic cubic function, $c(x) = x^3$, is an odd function because $-x^3 = (-x)^3$. Reflection across the x-axis reflects all the points above the graph to an equal distance below the graph and vice versa as shown with $-q(x)$ below.

x	$-q(x)$
–2	–4
–1	–1
0	0
1	–1
2	–4



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Dilation represents the third type of transformation. This type of transformation occurs when some number, c, is multiplied by the function. If $c > 1$, the function will be inflated or stretched. If $c < 1$, the function will be shrunk or compressed. This type of transformation can be denoted as $cq(x)$. Or, if $q(x) = x^2$ and $s(x) = cx^2$, then $s(x)$ is a dilation of $q(x)$. In effect, every y-value is multiplied by c as shown below with the table and graph of $4q(x)$.

x	$4q(x)$
–2	16
–1	4
0	0
1	4
2	16

In the example above, $c = 4$. In the example below, $c = \frac{1}{8}$:

x	$\frac{1}{8}q(x)$
–2	1/2
–1	1/8
0	0
1	1/8
2	1/2

An Informal Discussion continued. . .

In Sections 1.1-1.3, we discussed the monthly income of a paperboy who earns \$4.50 for every subscriber to whom he delivers the paper. We used the function below to describe the paperboy's monthly income paid by the newspaper.

$$p(x) = \$4.50x$$

Now, let's assume that our paperboy is so efficient that his subscribers leave him tips. Indeed, we will assume that he earns an average of two dollars per subscriber per month on tips. The paperboy's monthly income from tips is given below:

$$t(x) = \$2.00x$$

We have two monthly income functions for the paperboy, $p(x)$ and $t(x)$. The first, $p(x)$, represents the income paid by the newspaper. The second, $t(x)$, represents his tips. The sum of the two functions, denoted as $p + t$ or as $(p + t)(x)$, represents his total monthly income.

$$p + t = p(x) + t(x)$$
$$p + t = \$4.50x + \$2.00x$$
$$p + t = \$6.50x$$

Instruction: *Operations and Transformations*

Example 1
Operations with functions

Consider $f(x) = x^2 + x + 4$ and $g(x) = 12x$. Find the following:

A. $f + g$ B. fg C. $g - f$ D. $f - g$ E. $\frac{f}{g}$

A. The notation $f + g$ indicates the sum of the two functions.

$$f + g = \left(x^2 + x + 4\right) + \left(12x\right)$$
$$f + g = x^2 + 13x + 4$$

B. The notation fg indicates the product of the two functions.

$$fg = \left(x^2 + x + 4\right)12x$$
$$fg = 12x^3 + 12x^2 + 48x$$

C. The notation $g - f$ indicates the difference of g and f.

$$g - f = 12x - \left(x^2 + x + 4\right)$$
$$g - f = 12x - x^2 - x - 4$$
$$g - f = -x^2 + 11x - 4$$

D. The notation $f - g$ indicates the difference of f and g.

$$f - g = \left(x^2 + x + 4\right) - \left(12x\right)$$
$$f - g = x^2 - 11x + 4$$

E. The notation f/g indicates the quotient of f and g.

$$\frac{f}{g} = \frac{x^2 + x + 4}{12x},\ x \neq 0$$

Example 2
Composition of Functions

Given $M(x) = 4 - x^2$ and $N(x) = 6x$, show that composition of functions is not commutative.

Find $M \circ N(x)$.

$$M \circ N(x) = M(N(x)) = 4 - (6x)^2$$
$$M \circ N(x) = 4 - 36x^2$$

Find $N \circ M(x)$.

$$N \circ M(x) = N(M(x)) = 6(4 - x^2)$$
$$N \circ M(x) = 24 - 6x^2$$

Note that $M \circ N(x) \neq N \circ M(x)$.

$$4 - 36x^2 \neq 24 - 6x^2$$

Example 3
Composition of Functions

Given $T(x) = x^2 + 2x$ and $S(x) = 2x + 3$. A) Find $T \circ S(x)$. B) Evaluate $S(T(2))$.

A. Find $T \circ S(x)$.

$$T \circ S(x) = T(S(x)) = (2x+3)^2 + 2(2x+3)$$
$$T \circ S(x) = (2x+3)(2x+3) + 4x + 6$$
$$T \circ S(x) = 4x^2 + 12x + 9 + 4x + 6$$
$$T \circ S(x) = 4x^2 + 16x + 15$$

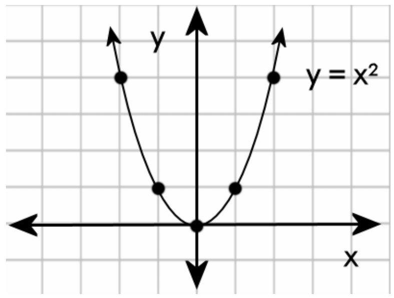
B. Find $T(2)$ and evaluate $S(T(2))$.

$$T(2) = (2)^2 + 2(2) = 8$$
$$S(8) = 2(8) + 3 = 16 + 3 = 19$$
$$\therefore S(T(2)) = 19$$

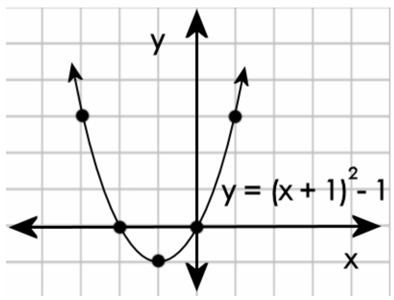
Example 4
Transformation of Functions

Given $f(x) = x^2$, graph $f(x+1)-1$.

Sketch the graph of *f*(*x*).



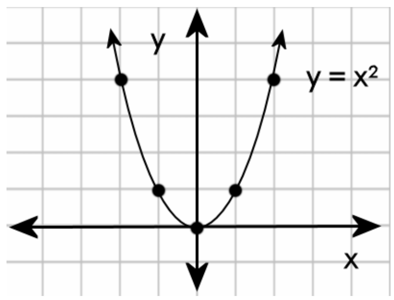
Sketch the graph of $f(x+1)-1$. Slide each point one unit to the left and one unit down.



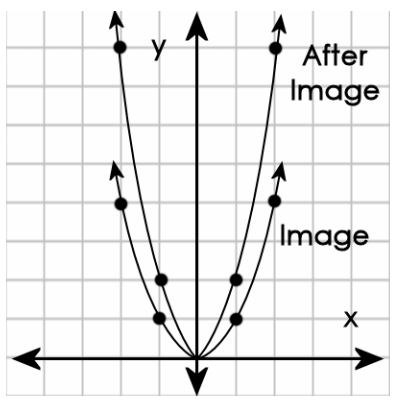
Example 5
Transformation of Functions

Given $f(x) = x^2$, graph both $f(x)$ and the dilation $2f(x)$ on the same Cartesian plane.

Sketch the graph of $f(x)$.



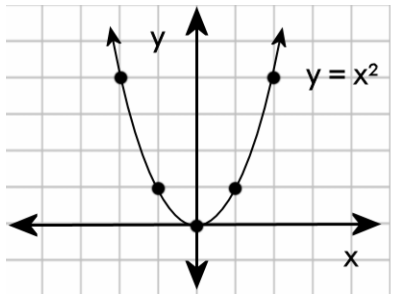
Sketch the dilation $2f(x)$ on the same x-y plane as the graph of $f(x)$. Take each point on $f(x)$ and double it's y-value.



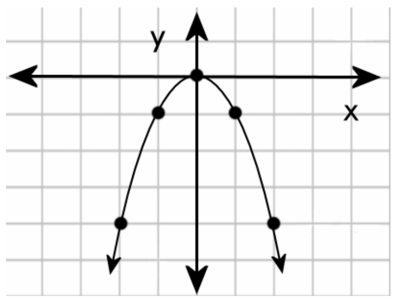
Example 6
Transformation of Functions

Given $f(x) = x^2$, graph $-f(x) + 1$.

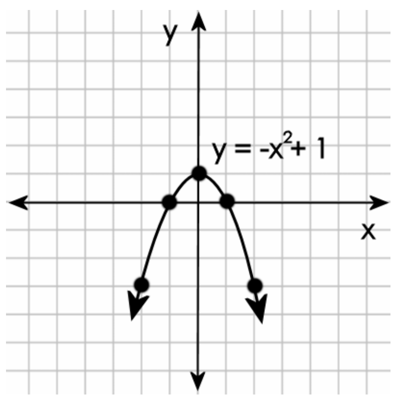
Sketch the graph of $f(x)$.



Sketch $-f(x)$, which reflects the graph over the x-axis, creating a mirror image.



Sketch $-f(x) + 1$ by shifting the graph of $-f(x)$ up one unit.



Practice Set 1.6A

#1 Given: $k(x)=\sqrt{x+5}$, evaluate $k(-1)$

#2 Given: $j(x)=6x$, evaluate $j(2)$

#3 Given: $m(x)=x^2-1$, evaluate $m(-5)$

#4 Given: $n(x)=5-x$, evaluate $n(-3)$

#5 Given: $p(x)=5x$, evaluate $p(a)$

#6 Given: $P(x)=\frac{5}{2}x^2$, evaluate $P(4)$

#7 Given: $q(x)=\frac{x}{x+1}$, evaluate $q(1)$

#8 Given: $f(x)=x^2+1$ and $g(x)=x+2$, evaluate $f(g(-2))$

#9 Given: $f(x)=x^2+1$ and $g(x)=x+2$, evaluate $f-g$

#10 Given: $f(x)=x^2+1$ and $g(x)=x+2$, evaluate $f+g$

#11 Given: $f(x)=x^2+1$ and $g(x)=x+2$, evaluate fg

#12 Given: $f(x)=x^2+1$ and $g(x)=x+2$, evaluate f/g

#13 Given: $f(x)=x^2+1$ and $g(x)=x+2$, evaluate $(f\circ g)(x)$

#14 Given: $G(x)=x^2+2$, evaluate $G(x+h)$

#15 Given: $G(x)=x^2+2$, evaluate $\frac{G(x+h)-G(x)}{h}$

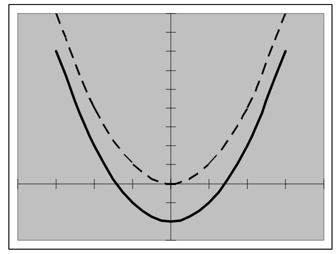
#16 Given: $r(x)=3x$ and $s(x)=8x$, evaluate $(r\circ s)(x)$

#17 Given: $v(x)=x^2+3$ and $w(x)=100x$, evaluate $(w\circ v)(x)$

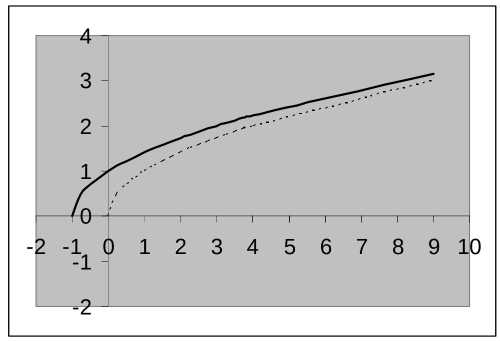
#1 2, #2 12, #3 24, #4 8, #5 5a, #6 40, #7 ½, #8 1, #9 $(f-g)(x)=x^2-x-1$, #10 $(f+g)(x)=x^2+x+3$, #11 $(fg)(x)=x^3+2x^2+x+2$,
#12 $\left(\frac{f}{g}\right)(x)=\frac{x^2+1}{x+2},\ x\neq -2$, #13 $(f\circ g)(x)=x^2+4x+5$, #14 $(G)(x+h)=x^2+2xh+h^2+2$,
#15 $\frac{G(x+h)-G(x)}{h}=2x+h$, #16 $(r\circ s)(x)=24x$, #17 $(w\circ v)(x)=100x^2+300$

Practice Set 1.6B

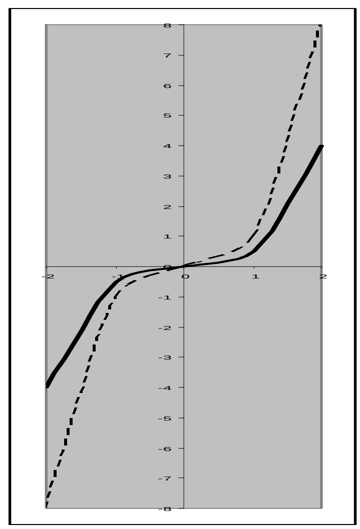
#1 If the dotted curve represents $y = x^2$,
write a function represented by the solid curve.



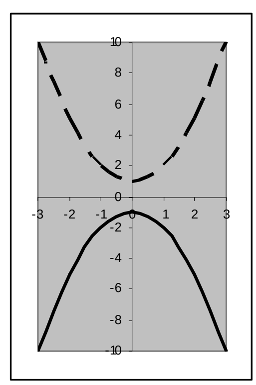
#2 If the dotted curve represents $y = \sqrt{x}$,
write a function represented by the solid curve.



#3 If the dotted curve represents $y = x^3$,
write a function represented by the solid curve.



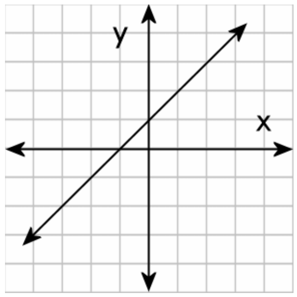
#4 If the dotted curve represents $y = x^2 + 1$,
write a function represented by the solid curve.

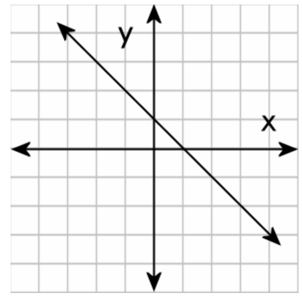


#1 $y = x^2 - 2$, #2 $y = \sqrt{x+1}$, #3 $y = \frac{1}{2}x^3$, #4 $y = -\left(x^2 + 1\right)$ or $y = -x^2 - 1$

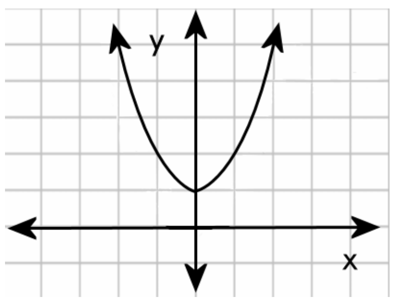
Practice Set 1.6C

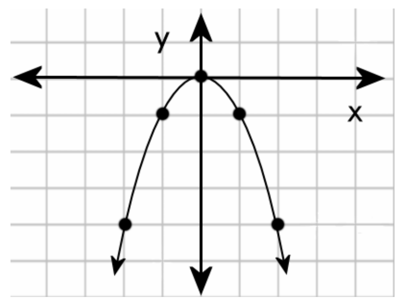
Consider the linear function $y = x$. For each graph below, describe the transformation.

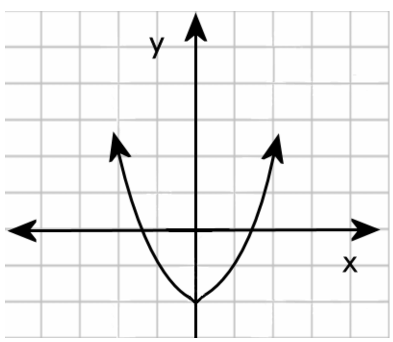
#1 

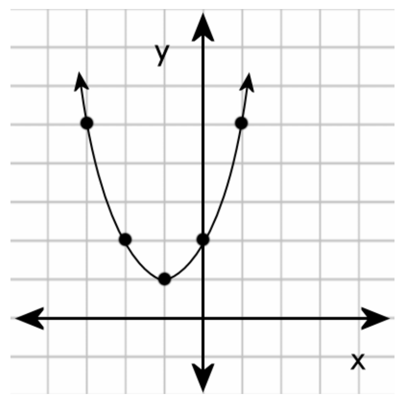
#2 

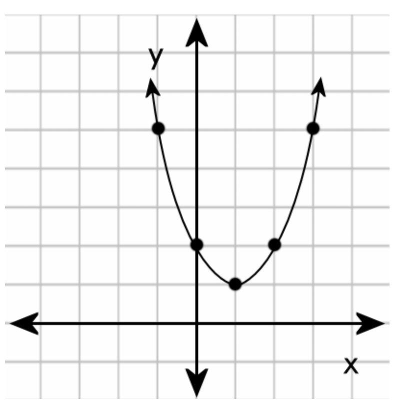
Consider the quadratic function $f(x) = x^2$. For each graph below, use functional notation to indicate the transformation.

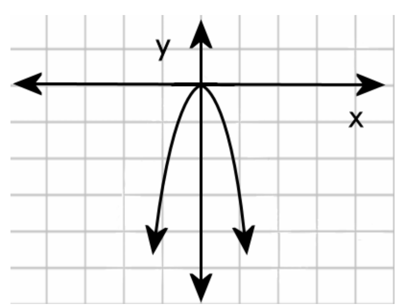
#3 

#4 

#5 

#6 

#7 

#8 

#1 shift up one, $y = x + 1$
#2 reflection across x-axis (or y-axis) and shift up one, $y = -x + 1$
#3 $f(x) + 1$
#4 $-f(x)$
#5 $f(x) - 2$
#6 $f(x+1) + 1$
#7 $f(x-1) + 1$
#8 $-3f(x)$

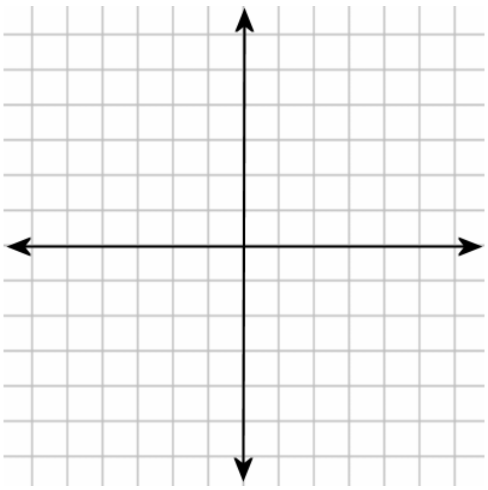
Study Exercise 1.6

Problems

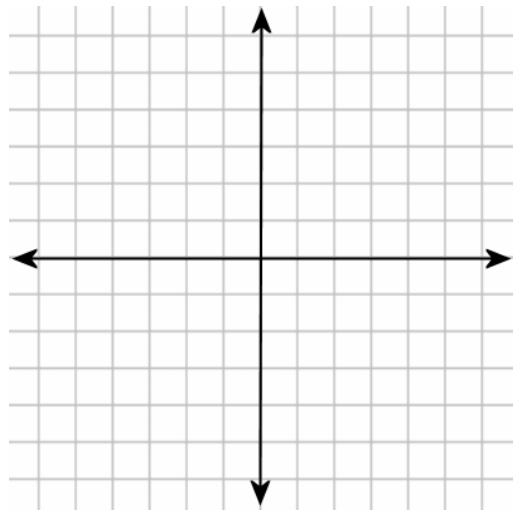
Consider functions *f* and *g*.

$$f(x) = x^2 \qquad g(x) = -1$$

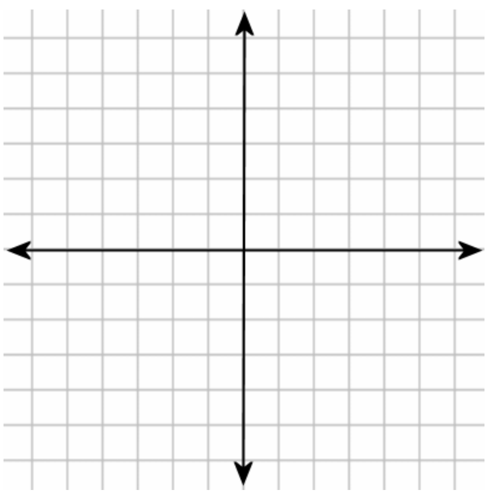
#1 Sketch the graph of $f(x+1)$.



#2 Sketch the graph of fg.



#3 Sketch the graph of $f + g$.



#4 Find $f \circ g$.