Lecture 2.4

College Algebra

Instruction: *Determining the Possible Rational Roots using the Rational Root Theorem*

Consider the theorem stated below.

> **Rational Root Theorem:** *If the rational number* $^b/_c$, *in lowest terms, is a root of the polynomial,* $P(x) = a_n x^n + a^{n-1} x^{n-1} + \ldots + a_1 x + a_0$ *with integer coefficients, then* b *must be an integer factor of* a_0 *and* c *must be an integer factor of* a_n.

The Rational Root Theorem tells us that any rational roots of a polynomial must be some quotient of the factors of the constant and leading coefficient. The Rational Root Theorem is an important tool in solving polynomials because it quickly yields a manageable set of *possible* rational roots. The Remainder Theorem (discussed in Section 2.5) determines whether a possible rational root is, in fact, a root.

The Rational Root Theorem states that any rational roots of a polynomial must be some quotient of the constant and leading coefficient. Consequently, placing the positive and negative factors of the constant over the positive and negative factors of the leading coefficient generates a list of possible rational roots of a polynomial. Using $f(x) = 2x^4 + 13x^3 + 13x^2 - 13x - 15$, for example, we can generate a list of possible rational roots thusly:

$$\frac{\textit{factors of } 15}{\textit{factors of } 2} = \frac{\pm 1, \pm 3, \pm 5, \pm 15}{\pm 1, \pm 2} = \pm \frac{1}{2} \pm 1, \pm \frac{3}{2}, \pm \frac{5}{2}, \pm 3, \pm 5, \pm \frac{15}{2}, \pm 15$$

Instruction: *The Rational Root Theorem*

Example 1
Determining *Possible* Rational Roots

Consider $w(x) = x^2 - 5$. Is two a possible root of $w(x)$?

According to the Rational Root Theorem, the possible rational roots of a polynomial are ratios of factors of the constants to the factors of the leading coefficient. Since two is not a factor of the constant, it cannot be a rational root of $w(x)$.

Example 2
Determining *Possible* Rational Roots

Consider $p_1(x) = x^2 + 5x - 6$ and $p_2(x) = x^2 + 6x + 9$. Which function $p_1(x)$ or $p_2(x)$ has the greatest number of *possible* rational roots according to the Rational Root Theorem?

According to the Rational Root Theorem, the possible rational roots of a polynomial function are ratios of factors of the constants to the factors of the leading coefficient. Since both functions have the same leading coefficient, the polynomial function whose constant has the greater number of distinct factors will have the greatest number of possible rational roots according to the Rational Root Theorem. The constant of $p_1(x)$ has four factors (1, 2, 3, 6) while the constant of $p_2(x)$ only has three factors (1, 3, 9); therefore, $p_1(x)$ will have more possible rational roots according to the Rational Root Theorem. (Incidentally, the *N* Roots Theorem from Section 2.3 indicates that both polynomials have the same number of not necessarily distinct roots since the degrees of the polynomial are equal.)

Example 3
Determining *Possible* Rational Roots

Consider $f(x) = x^2 + 5x + 4$. Is the following statement true or false?

Statement: The Rational Root Theorem guarantees that 4 is a root of $f(x)$.

The Rational Root Theorem provides a means for determining *possible* rational roots. It does not guarantee that any number is a root of any polynomial. The statement is false.

Example 4
Determining *Possible* Rational Roots

Consider $Z(x) = -6 + 5x + 3x^3$. List the *possible* rational roots according to the Rational Root Theorem.

Note the constant, –6, and the leading coefficient, 3. List the factors of the constant: 1, 2, 3, 6. List the factors of the leading coefficient: 1, 3. Write all the possible ratios of factors of the constant to factors of the leading coefficient.

$$\frac{1}{1}, \frac{1}{3}, \frac{2}{1}, \frac{2}{3}, \frac{3}{1}, \frac{3}{3}, \frac{6}{1}, \frac{6}{3}$$

Reduce each ratio.

$$1, \frac{1}{3}, 2, \frac{2}{3}, 3, 1, 6, 2$$

Eliminate redundancies.

$$1, \frac{1}{3}, 2, \frac{2}{3}, 3, 6$$

Include the opposites.

$$1, -1, \frac{1}{3}, -\frac{1}{3}, 2, -2, \frac{2}{3}, -\frac{2}{3}, 3, -3, 6, -6$$

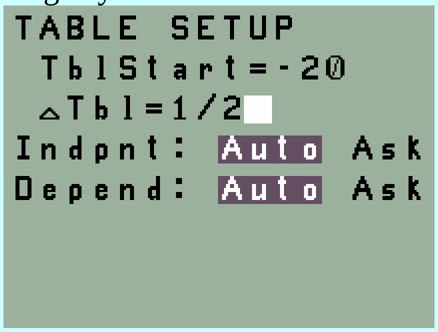
Example 5
Determining *Possible* Rational Roots

Consider $T(x) = 2x^2 + 19x + 9$. Is the following statement true or false?
Statement: According to the Rational Root Theorem, $1/2$ is a possible rational root of $T(x)$.

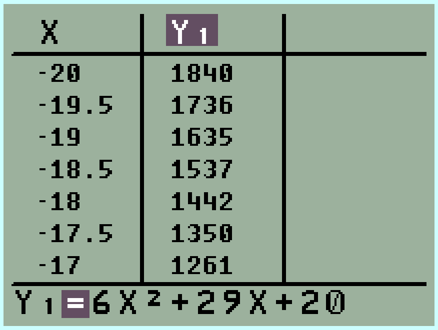
According to the Rational Root Theorem, the possible rational roots of a polynomial function are ratios of factors of the constants to the factors of the leading coefficient. Since 1 is a factor of the constant and 2 is a factor of the leading coefficient, the ratio of 1 to 2 is a possible rational root of the function according to the Rational Root Theorem. The statement is true.

Example 6
Determining *Possible* Rational Roots

Consider $g(x) = 6x^2 + 29x + 20$. Judicious uses his table set up menu to begin a table at negative twenty and to change by halves as shown below.



Accordingly, Judicious has a table in his graphing calculator that looks like the one below.



Judicious knows that all the roots of $g(x)$ are real and rational. He plans to scroll down the column of x-values looking for zeros in the column of y-values. Will Judicious find the roots of $g(x)$ assuming they are rational?

The Rational Root Theorem states that the possible rational roots of a polynomial function are ratios of factors of the constants to the factors of the leading coefficient. Since the factors of the leading coefficient include 3 and 6, it is possible that $g(x)$ has rational roots that are fractions with six and or three in the denominator. The table set up by Judicious starts at an integer and changes by halves; therefore, it will not land on thirds or sixths. Consequently, Judicious may not necessarily find the rational roots of the polynomial; even though, he judiciously started his table at the lowest possible number that could be a rational root according to the Rational Root Theorem.

Practice Set 2.4

Identify all the possible rational roots for the following polynomial functions using the Rational Root Theorem.

#1 $f(x) = -x^3 - 6x^2 + 13x + 42$

#2 $D(x) = 12x^2 - 23x - 2$

#3 $C(x) = x^3 + 4x^2 - 3x - 12$

#4 $p(x) = -2x^2 - 121x - 60$

#5 $r(x) = x^4 - 5x^2 + 4$

#1 ±1, ±2, ±3, ±6, ±7, ±14, ±21, ±42
#2 $\pm {}^{1}/_{12}$, $\pm {}^{1}/_{6}$, $\pm {}^{1}/_{4}$, $\pm {}^{1}/_{3}$, $\pm {}^{1}/_{2}$, $\pm {}^{2}/_{3}$, ±1, ±2
#3 ±1, ±2, ±3, ±4, ±6, ±12
#4 $\pm {}^{1}/_{2}$, ±1, $\pm {}^{3}/_{2}$, ±2, $\pm {}^{5}/_{2}$, ±3, ±4, ±5, ±6, $\pm {}^{15}/_{2}$, ±10, ±12, ±15, ±20, ±30, ±60
#5 ±1, ±2, ±4

Assignment 2.4

Problems

Identify all the possible rational roots for the following polynomial functions using the Rational Root Theorem.

1 $g(x) = x^4 - 13x^2 + 36$

2 $P(x) = 12x^3 + 24x^2 - 12x - 24$

Hint: Factoring the greatest common factor simplifies the list of possible rational roots.

3 $m(x) = 6x^3 + 37x^2 + 37x + 10$

4 $h(x) = 6 + 4x - 9x^2 - 6x^3 + 3x^4 + 2x^5$

College Algebra

Instruction: ***The Remainder Theorem***

This section introduces the division algorithm, which when coupled with the remainder theorem provides an algorithm for identifying the roots of a polynomial.

Consider two polynomial functions *P*(*x*) with degree n_1 and *d*(*x*) with degree n_2 where $0 < n_2 \leq n_1$. The quotient *P*(*x*)/*d*(*x*) can be expressed as *P*(*x*)/*d*(*x*) = *Q*(*x*) + *R*(*x*)/*d*(*x*) where *P*(*x*) is the dividend, *d*(*x*) the divisor, *Q*(*x*) the quotient without the remainder, and *R*(*x*) is the remainder with a degree less than the divisor *d*(*x*). The quotient of *P*(*x)* and *d*(*x*) is below.

$$\frac{P(x)}{d(x)} = Q(x) + \frac{R(x)}{d(x)}$$

Multiplying both sides by *d*(*x*) yields:.

$$d(x)\frac{P(x)}{d(x)} = d(x)\left(Q(x) + \frac{R(x)}{d(x)}\right)$$
$$P(x) = Q(x)\cdot d(x) + R(x).$$

If *d*(*x*) is linear so that its degree equals 1, then the degree of *R*(*x*) must be zero and *R*(*x*) equals a constant, *R*. Allowing *d*(*x*) = *mx* + *b* yields:

$$P(x) = Q(x)\cdot(mx+b) + R\,.$$

Letting $x = -\frac{b}{m}$ arrives at the theorem stated in the box below.

$$P\left(-\frac{b}{m}\right) = Q\left(-\frac{b}{m}\right)\cdot\left(m\left(-\frac{b}{m}\right)+b\right)+R$$
$$P\left(-\frac{b}{m}\right) = Q\left(-\frac{b}{m}\right)\cdot(-b+b)+R$$
$$P\left(-\frac{b}{m}\right) = Q\left(-\frac{b}{m}\right)\cdot(0)+R$$
$$P\left(-\frac{b}{m}\right) = R$$

Remainder Theorem: *If* R *is the remainder after dividing the polynomial* P(x) *by* mx + b*, then* $P\left(-\frac{b}{m}\right) = R\,.$

Lecture 2.5

The Remainder Theorem tells us that the remainder of the division algorithm is the *y*-value coordinated with the divisor. To illustrate, consider $w(x) = x^2 + 6x + 8$. Now consider the synthetic division using "1" as the divisor and the coefficients of $w(x)$ as the dividend.

$$\begin{array}{r|rrr} 1 & 1 & 6 & 8 \\ & \downarrow & 1 & 7 \\ \hline & 1 & 7 & 15 \end{array}$$

According to the Remainder Theorem, the remainder in the synthetic division is the *y*-value coordinated with the divisor. Consequently, according to the Remainder Theorem, $w(1) = 15$. This theorem is important because it reveals a method for finding roots discussed below.

Instruction: *Finding all the Roots of Cubic Functions With Only One Rational Root*

The Real Roots and Odd-Degree Polynomial Theorem stated below tells us that odd-degree polynomials have at least one real root.

Real Roots and Odd-Degree Polynomial Theorem: *A polynomial of odd degree with real coefficients always has at least one real root.*

Finding one rational root of a cubic with synthetic division makes it possible to find all the roots. For example, consider $f(x) = x^3 + 11x^2 + 12x + 2$. The Rational Root Theorem (discussed in Section 2.4) identifies –1 as a possible rational root. Synthetic division with –1 yields a remainder of zero as seen below.

$$\begin{array}{r|rrrr} -1 & 1 & 11 & 12 & 2 \\ & & -1 & -10 & -2 \\ \hline & 1 & 10 & 2 & 0 \end{array}$$

According to the Remainder Theorem, $f(-1) = 0$, thus –1 must be a root of $f(x)$.

The division algorithm verified that –1 is a root of the polynomial since it yielded a zero in the final column. Furthermore, the numbers on the bottom row represent the coefficients to a factor of $f(x)$. Dividing successfully by –1, proves that $x + 1$ is a linear root of $f(x)$. Since $f(x)$ is a cubic (3^{rd} degree) polynomial and the division algorithm factored out a linear factor, the factor represented on the bottom row must be a quadratic (2^{nd} degree) factor. Thus, $x^2 + 10x + 2$ is a quadratic factor of $f(x)$. The quadratic formula will yield the final two roots.

Recall that for equations $ax^2 + bx + c = 0$, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$x = \frac{-10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$
$$x = \frac{-10 \pm \sqrt{92}}{2}$$
$$x = \frac{-10 \pm \sqrt{4 \cdot 23}}{2}$$
$$x = \frac{-10 \pm 2\sqrt{23}}{2}$$
$$x = \frac{2\left(-5 \pm 1\sqrt{23}\right)}{2}$$
$$x = -5 \pm \sqrt{23}.$$

So, the roots of $f(x)$ are $-1,\ -5+\sqrt{23},$ and $-5-\sqrt{23}$. Furthermore, if we apply the Factor Theorem, $f(x)$ can be written as a product of linear factors as seen below.

$$f(x) = (x+1)\left(x+5+\sqrt{23}\right)\left(x+5-\sqrt{23}\right)$$

Example Exercises 2.5

Instruction: *Remainder Theorem*

Example 1
Remainder Theorem

Consider $f(x) = 2x^4 + 3x^3 - 25x^2 + 8x + 18$. Use the remainder theorem to show that $f(3) = 60$

Divide synthetically by three.

$$\begin{array}{r|rrrrr} 3 & 2 & 3 & -25 & 8 & 18 \\ & & 6 & 27 & 6 & 42 \\ \hline & 2 & 9 & 2 & 14 & 60 \end{array}$$

According to the remainder theorem, $f(3) = R$, where R is the remainder in the synthetic division algorithm; thus, $f(3) = 60$.

Example 2
Remainder Theorem

Consider $g(x) = 2x^3 + 9x^2 - 7x - 4$. Show that $g(x)$ has three real roots, one rational and two irrational.

Divide synthetically by possible rational roots until a zero appears in the last entry of the quotient row.

$$\begin{array}{r|rrrr} 1 & 2 & 9 & -7 & -4 \\ & & 2 & 11 & 4 \\ \hline & 2 & 11 & 4 & 0 \end{array}$$

The zero indicates that the divisor, 1, is a root, so $g(x)$ has at least one rational root. Interpret the quotient row with a zero remainder as a quadratic factor of the polynomial: $2x^2 + 11x + 4$. Use the discriminant to determine if the roots of the quadratic factor are real or complex and, if real, rational or irrational.

$$2x^2 + 11x + 4 = 0$$
$$b^2 - 4ac = (11)^2 - 4(2)(4) = 89$$

Since the discriminant equals a positive number, the two roots of the quadratic are real. Since the discriminant is not a perfect square, the two roots are irrational.

The function has one rational root and two irrational roots.

Example 3
Finding the Roots of a Cubic

Consider $h(x) = 2x^3 + 15x^2 - 29x - 18$. Find all the roots of $h(x)$.

Divide synthetically by possible rational roots until a zero appears in the last entry of the quotient row.

$$\begin{array}{r|rrrr} 2 & 2 & 15 & -29 & -18 \\ & & 4 & 38 & 18 \\ \hline & 2 & 19 & 9 & 0 \end{array}$$

The zero indicates that the divisor, 2, is a root. Interpret the quotient row with a zero remainder as a quadratic factor of the polynomial: $2x^2 + 19x + 9$. Set the quadratic factor equal to zero and solve to find the last two roots.

$$2x^2 + 19x + 9 = 0$$
$$(2x+1)(x+9) = 0$$
$$2x + 1 = 0 \quad x + 9 = 0$$
$$2x = -1 \quad x = -9$$
$$x = -\frac{1}{2}$$

The roots of $h(x)$ are 2, –9, and $-1/2$.

Example 4
Finding the Roots of a Cubic

Consider $y(x) = 2x^3 + 9x^2 - 4x - 4$. Write $y(x)$ as a product of its linear factors.

Find the roots and use the factor theorem to identify the linear factors from the roots. Keep in mind the polynomial has a leading coefficient of 2. To find the roots, start dividing synthetically by possible rational roots until a zero appears in the last entry of the quotient row.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & 9 & -4 & -4 \\ & & -1 & -4 & 4 \\ \hline & 2 & 8 & -8 & 0 \end{array}$$

The zero indicates that the divisor, $-1/2$, is a root. Interpret the quotient row with a zero remainder as a quadratic factor of the polynomial: $2x^2 + 8x - 8$. Set the quadratic factor equal to zero and solve to find the last two roots. Factor the greatest common factor, but recognize it as a constant factor of the polynomial.

$$2x^2 + 8x - 8 = 0$$
$$2\left(x^2 + 4x - 4\right) = 0$$
$$a = 1,\ b = 4,\ c = -4$$
$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(-4)}}{2(1)}$$
$$x = \frac{-4 \pm \sqrt{16 - 4(-4)}}{2}$$
$$x = \frac{-4 \pm \sqrt{32}}{2}$$
$$x = \frac{-4 \pm \sqrt{16 \cdot 2}}{2}$$
$$x = \frac{-4 \pm 4 \cdot \sqrt{2}}{2}$$
$$x = \frac{-4}{2} \pm \frac{4\sqrt{2}}{2}$$
$$x = -2 \pm 2\sqrt{2}$$

The roots are $x = -2 + 2\sqrt{2}$, $x = -2 - 2\sqrt{2}$, and $-1/2$. Using the Factor Theorem, the linear factors are $\left(x + 2 - 2\sqrt{2}\right)\left(x + 2 + 2\sqrt{2}\right)\left(x + 1/2\right)$. Since the polynomial has a leading coefficient of 2, the polynomial has a constant factor of 2 and can be written as a product of factors as below.

$$y(x) = 2\left(x + 2 - 2\sqrt{2}\right)\left(x + 2 + 2\sqrt{2}\right)\left(x + 1/2\right).$$

Practice Set 2.5

Identify all the roots of the cubic functions below.

#1 $f(x) = x^3 + 3x^2 - x - 3$

#2 $g(x) = x^3 - 2x^2 + 5x - 24$

#3 $h(x) = x^3 + 4x^2 - 11x - 30$

#4 $j(x) = x^3 + 9x^2 + 5x + 45$

#5 $k(x) = 2x^3 + 10x^2 + 14x + 4$

#6 $p(x) = 2x^3 + 11x^2 + 13x + 4$

#7 $P(x) = 3x^3 - 14x^2 - 5x$

#8 $P(x) = -12x^3 + 29x^2 - 10x$

#1 $x = 1, x = -1, x = -3$

#2 $x = 3,\ x = \dfrac{-1 \pm i\sqrt{31}}{2}$

#3 $x = -5, x = -2, x = 3$

#4 $x = -9,\ x = i\sqrt{5},\ x = -i\sqrt{5}$

#5 $x = -2,\ x = \dfrac{-3 \pm \sqrt{5}}{2}$

#6 $x = -\frac{1}{2}, x = -1, x = -4$

#7 $x = 0, x = 5, x = -\frac{1}{3}$

#8 $x = 0, x = 2, x = \frac{5}{12}$

Assignment 2.5

Problems

Identify all the roots of the cubic functions below.

1 $g(x) = x^3 - 5x^2 - 23x + 3$

2 $h(x) = 2x^3 - 5x^2 - 4x + 3$

3 $P(x) = x^3 + 6x^2 + 6x + 1$

4 $L(x) = x^3 + 3x^2 + 5x + 3$

College Algebra

Instruction: *Finding all the Roots of Polynomials*

This lecture extends the topics discussed in Section 2.5, beginning with a stated theorem.

Upper Bound Theorem: *If* P(x) *is an* n*-degree polynomial with a non-zero constant and n > 0, a synthetic division quotient of* P(x)/(x – k) *of all the same sign (zero taking the sign of the term before it) where* k *is a positive real number indicates that* k *is an upper bound--that is,* P(x) *has no roots greater than* k*.*

The Upper Bound Theorem tells us that if a positive divisor yields a quotient in the synthetic division algorithm with all positive or all negative entries (with zero taking the sign of the entry before it), then the divisor is an upper bound, meaning the polynomial represented by the dividend has no roots greater than the divisor.

Lower Bound Theorem: *If* P(x) *is an* n*-degree polynomial with a non-zero constant and* n *> 0, a synthetic division quotient for* P(x)/(x – k) *of alternating signs (with zero taking the opposite sign of the term before it) where* k *is a negative real number indicates that* k *is a lower bound--that is,* P(x) *has no roots less than* k*.*

The Lower Bound Theorem tells us that if a negative divisor yields a quotient in the synthetic division algorithm with entries whose signs alternate from positive to negative or from negative to positive (with zero taking the opposite sign of the entry before it), then the divisor is a lower bound, meaning the polynomial represented by the dividend has no roots less than the divisor.

The division algorithm with the Rational Root Theorem and the Remainder Theorem makes it possible to find all the roots of an n-degree polynomial if the polynomial has at least $n - 2$ rational roots. For example, consider $f(x) = 10x^5 + 19x^4 - 29x^3 - 74x^2 - 44x - 8$. The Rational Root Theorem identifies the following list of possible rational roots:

$$\pm\left(\frac{1}{10}, \frac{1}{5}, \frac{2}{5}, \frac{1}{2}, \frac{4}{5}, 1, \frac{8}{5}, 2, 4, 8\right).$$

Descartes' Rule of Signs tells us that there is only one positive real root because there is only one sign change in the polynomial when it is written in descending order. Since there are five total roots and only one is positive there are at most four negative real roots. Instead of four negative real roots, there could also be only two negative real roots or none at all after considering that some roots may be non-real, complex roots, which always appear in pairs.

Choosing which number from the list of possible rational roots is a guessing game, but like all guessing games there are strategies that produce greater probabilities of success. For this polynomial, using a negative number from the list of possible rational roots is more likely to yield a root since there are possibly four negative roots. Thus:

–4	10	19	–29	–74	–44	–8
		–40	84	–220	1176	–4528
	10	–21	55	–294	1132	–4536

Since negative four yielded a non-zero number in the last column, negative four is not a root. Indeed, negative four represents a lower bound for negative roots since the quotient alternates signs (see Lower Bound Theorem). Consequently, negative eight is eliminated as a possible rational root. Thus, a negative number closer to zero should be chosen:

–1	10	19	–29	–74	–44	–8
		–10	–9	38	36	8
	10	9	–38	–36	–8	0

Since negative one yielded a zero in the last column, negative one is a root. Now, synthetic division should be performed on the quartic (4th-degree) quotient represented by the last row.

–1	10	19	–29	–74	–44	–8
		–10	–9	38	36	8
–2	10	9	–38	–36	–8	0
		–20	22	32	8	
	10	–11	–16	–4	0	
–1		–10	21	–5		
	10	–21	5	–9		

In the above division algorithms, negative two proved to be a root by yielding a zero in the last column when it was divided into the quartic quotient. Negative one was then used as the divisor a second time. This is a valid attempt since roots are often repeated. If negative one had worked a second time, it would have yielded a zero in the last column when it was divided into cubic factor represented by the quotient corresponding with negative two (as shown above). Since negative one did not yield a zero, the division algorithm must be returned to the last row with a zero in the final column (since rows with zeroes in the last column represent factors of the polynomial):

$-\frac{1}{2}$	10	–11	–16	–4
		–5	8	4
	10	–16	–8	0

This last algorithm yields the third root. Negative one-half, negative one, and negative two are all roots of the polynomial. Remember that the division algorithm discovers roots, but it also factors linear factors from the polynomial. Thus, each time synthetic division is performed—first into the polynomial then into the reduced factors represented by final rows with zeroes in the last column—the polynomial is reduced. Reducing a 5th-degree polynomial three times, reduces it to a 2nd-degree factor. Consequently, the last row above represents a quadratic factor: $10x^2 - 16x - 8$. The quadratic factor, of course, contains two roots. Setting the quadratic equal to zero and solving either by factoring, completing the square, or the quadratic formula will yield the final two roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(10)(-8)}}{2(10)}$$

$$x = \frac{16 \pm \sqrt{256 + 320}}{20}$$

$$x = \frac{16 \pm \sqrt{576}}{20}$$

$$x = \frac{16 \pm 24}{20}$$

$$x = \frac{16 + 24}{20} \quad and \quad x = \frac{16 - 24}{20}$$

$$x = \frac{40}{20} \quad and \quad x = \frac{-8}{20}$$

$$x = 2 \quad and \quad x = -\frac{2}{5}$$

Thus, the roots of *f*(*x*) are $-1, -2, -\frac{1}{2}, -\frac{2}{5}, 2$. All the roots of *f*(*x*) are rational. This will not always be the case. Sometimes the quadratic formula will yield two irrational roots or, possibly, two complex roots (when a negative appears under the square root). Of course, many polynomials will have more than one pair of irrational or complex roots. For our considerations, however, polynomials will have at most one pair of irrational or complex roots.

Instruction: *Finding the Roots of Polynomial Functions*

Example 1
Identifying the Upper and Lower Bounds

Consider $f\left(x\right) = 2x^5 + x^4 - 5x^3 - 10x^2 - 2x + 4$. Find the largest integer lower bound and the smallest integer upper bound on the roots of *f*(*x*) according to the Lower Bound Theorem and the Upper Bound Theorem.

Divide synthetically by negative integers starting with negative one until entries with alternating signs appear in the quotient row (remember that zero takes the sign of the entry before it).

-1	2	1	-5	-10	-2	4
		-2	1	4	6	-4
	2	-1	-4	-6	4	0

-2	2	1	-5	-10	-2	4
		-4	6	-2	24	-44
	2	-3	1	-12	22	-40

Since the quotient row is a row of entries with alternating signs, negative two is a lower bound on the roots of *f*(*x*), meaning *f*(*x*) has no roots smaller (further left) than negative two.

Divide synthetically by positive integers starting with positive one until entries with all the same signs appear in the quotient row (remember that zero takes the sign of the entry before it).

1	2	1	-5	-10	-2	4
		2	3	-2	-12	-14
	2	3	-2	-12	-14	-10

2	2	1	-5	-10	-2	4
		4	10	10	0	-4
	2	5	5	0	-2	0

3	2	1	-5	-10	-2	4
		6	21	48	114	336
	2	7	16	38	112	340

The largest integer lower bound is -2. The smallest integer upper bound is 3.

Example 2
Finding the Roots of Polynomial Functions

> Consider $f(x) = 2x^5 + x^4 - 5x^3 - 10x^2 - 2x + 4$.
>
> I. Find the roots of *f*(*x*). II. Write *f*(*x*) as a product of linear factors.

According to Descartes' Rule of Signs, *f*(*x*) has at most two positive roots because it has two sign changes in expanded form and descending order. Since the degree is five, *f*(*x*) has at most three negative roots and at least one negative root. Divide synthetically by possible rational roots until a zero appears in the last entry of the quotient row. When appropriate, use the Upper Bound Theorem and the Lower Bound Theorems to help reduce the number of necessary attempts.

$$\begin{array}{r|rrrrrr} -1 & 2 & 1 & -5 & -10 & -2 & 4 \\ & & -2 & 1 & 4 & 6 & -4 \\ \hline & 2 & -1 & -4 & -6 & 4 & 0 \end{array}$$

The zero indicates that the divisor, –1, is a root. Interpret the quotient row as a factor of the polynomial and continue dividing synthetically looking for zero remainders.

$$\begin{array}{r|rrrrr} 2 & 2 & -1 & -4 & -6 & 4 \\ & & 4 & 6 & 4 & -4 \\ \hline & 2 & 3 & 2 & -2 & 0 \end{array}$$

The zero indicates that the divisor, 2, is a root. Interpret the quotient row as a factor of the polynomial and continue dividing synthetically looking for zero remainders.

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & 3 & 2 & -2 \\ & & 1 & 2 & 2 \\ \hline & 2 & 4 & 4 & 0 \end{array}$$

The zero indicates that the divisor, $1/2$, is a root. Interpret the quotient row as a quadratic factor of the polynomial: $2\left(x^2 + 2x + 2\right)$. Set the quadratic factor equal to zero and solve.

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(2)}}{2(2)} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm 2i}{4} = \frac{\cancel{2}(-1 \pm i)}{\cancel{4}} = -1 \pm i$$

I. The roots of *f*(*x*) are –1, 2, $1/2$, $-1+i$, and $-1-i$.

II. $f(x) = 2(x+1)(x-2)(x-1/2)(x+1-i)(x+1+i)$

Practice Set 2.6

Identify all the roots of the polynomial functions.

#1 $P(x) = x^4 - \frac{33}{10}x^3 + \frac{23}{10}x^2 + \frac{3}{5}x$

HINT FOR #1: Factor out the greatest common factor; then use synthetic division to factor the cubic factor.

#2 $G(x) = x^4 - 2x^3 - 14x^2 + 30x + 9$

#3 $H(x) = -3x^5 + 2x^4 - 6x^3 - 20x^2 + x + 10$

#4 $Q(x) = 4x^4 + 4x^3 + 49x^2 + 64x - 240$

#5 $K(x) = x^3 - 1$

ANSWERS

#1

$x = -1/5$, $x = 0$, $x = 3/2$, $x = 2$

#2

$x = -2 \pm \sqrt{3}$, $x = 3$ [double multiplicity, meaning the polynomial has a repeated linear factor, $(x - 3)^2$]

#3

$x = 2/3$, $x = -1$ [double multiplicity, the polynomial has a repeated linear factor, $(x + 1)^2$], $x = 1 \pm 2i$

#4

$x = -5/2$, $x = 3/2$, $x = \pm 4i$

#5

$x = 1$, $x = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$

Practice Set 2.6_Supplemental

Identify all the roots (real and complex) of the polynomial functions.

#1 $p(x) = x^4 - 3x^3 - 4x^2$

#2 $p(x) = 3x^5 + 7x^4 - 103x^3 - 143x^2 + 504x + 180$

#3 $y(x) = x^3 + x^2 - 12x$

#4 $Y(x) = x^4 - 2x^3 - 11x^2 + 12x + 36$

#5 $m(x) = 10x^4 + 23x^3 - 296x^2 - 399x + 270$

#6 $M(x) = x^4 - 7x^3 + 2x^2 + 64x - 96$

#7 $F(x) = -8x^4 + 4x^3 + 18x^2 + 11x + 2$

#8 $f(x) = x^4 + 13x^3 + x^2 - 143x - 12$

#9 $\delta(x) = -2x^5 - 8x^4 - 12x^3 - 16x^2 + 14x + 24$

#10 $\lambda(x) = x^7 + 6x^6 + 12x^5 + 8x^4$

ANSWERS

#1 $x = -1$, $x = 0$ [double multiplicity], $x = 4$
#2 $x = -6$, $x = -3$, $x = -\frac{1}{3}$, $x = 2$, $x = 5$
#3 $x = -4$, $x = 0$, $x = 3$
#4 $x = -2$ [double multiplicity], $x = 3$ [double multiplicity]
#5 $x = -6$, $x = -\frac{9}{5}$, $x = \frac{1}{2}$, $x = 5$
#6 $x = -3$, $x = 2$, $x = 4$ [double multiplicity]
#7 $x = -\frac{1}{2}$ [triple multiplicity], $x = 2$
#8 $x = -4$, $x = 3$, $x = -6 - \sqrt{35}$, $x = -6 + \sqrt{35}$
#9 $x = -3$, $x = -1$, $x = 1$, $x = \frac{-1 - i\sqrt{15}}{2}$, $x = \frac{-1 + i\sqrt{15}}{2}$
#10 $x = -2$ [triple multiplicity], $x = 0$ [quadruple multiplicity]

Assignment 2.6

Problems

Identify all the roots of the polynomial functions.

1 $g(x) = x^5 - 6x^4 - 9x^3 + 54x^2 + 20x - 120$

2 $h(x) = -x^4 - 3x^3 - 4x^2 - 3x - 1$

3 $P(x) = x^5 + x^4 - 86x^3 - 86x^2 + 405x + 405$

4 $c(x) = -2x^3 - 6x^2 + 4x + 12$

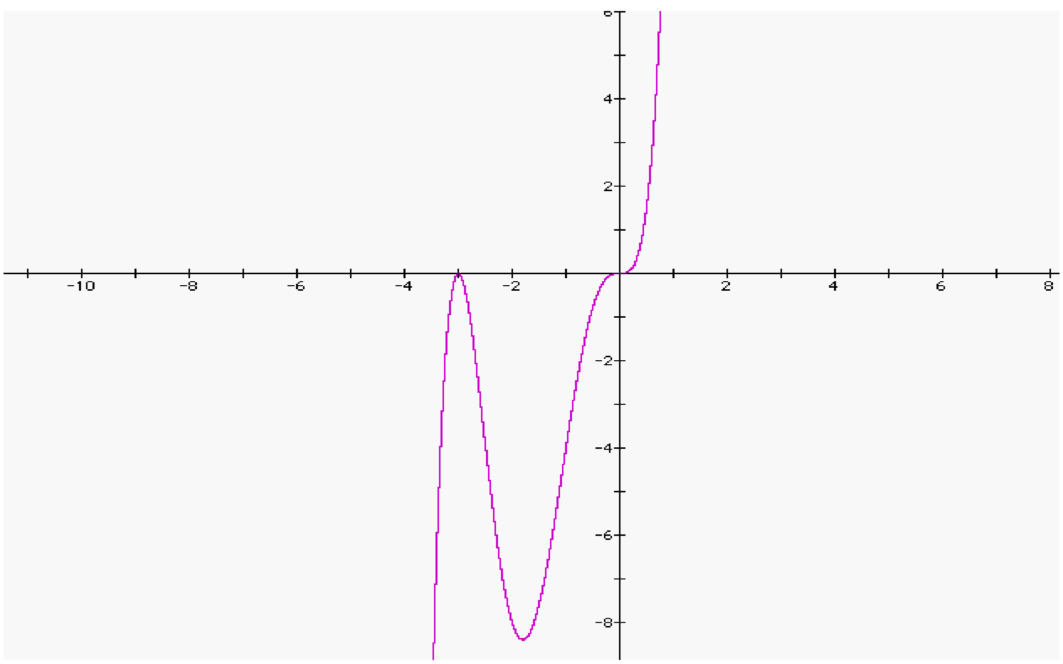
College Algebra

Instruction: *Graphing Multiplicity Behavior*

Consider $f(x) = x^5 + 6x^4 + 9x^3$. The function can be factored first by factoring out the greatest common factor and then by factoring the resulting trinomial:

$$f(x) = x^5 + 6x^4 + 9x^3$$
$$f(x) = x^3\left(x^2 + 6x + 9\right)$$
$$f(x) = x^3(x+3)(x+3)$$

Thus, the roots of the polynomial are zero with a multiplicity of three and negative three with a multiplicity of two. The graph of a polynomial with repeated roots behaves in predictable ways near the *x*-axis. As multiplicity increases the graph will flatten to a more pronounced degree near the repeated root. Also, odd multiplicity indicates that the graph will cut through the *x*-axis at the root, and even multiplicity indicates that the graph will bounce off the *x*-axis at the root. Thus, the graph of *f*(*x*) will appear accordingly:

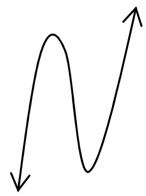


Instruction: *Multiplicity of Roots*

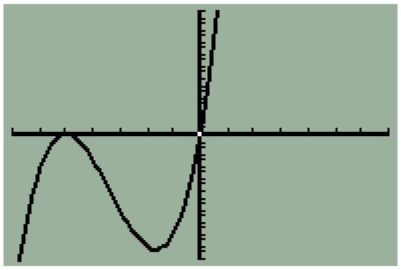
Example 1
Graphing Polynomials with Multiplicity of Roots

Consider $f(x) = x(x+5)^2$. Graph the polynomial. Show proper near *x*-axis behavior.

Note that the polynomial has three linear factors, so it is a third-degree polynomial and has three roots and opposite end behavior. Since the leading coefficient is positive, the polynomial rises on the right.



Use the Factor Theorem to recognize the roots as zero and negative five. Note that the factor $(x+5)$ is repeated an even number of times. This repetition is called multiplicity. Whenever multiplicity is even--that is, whenever the root is repeated an even number of times--the graph will "bounce off" the *x*-axis as shown below.



Example 2
Finding Multiplicity

Consider $d\left(t\right)=t^5+9t^4+30x^3+46x^2+33x+9$. Show that $d(t)$ has multiplicity of roots.

Use Descartes' Rule of Signs to note that the polynomial does not have any positive roots since it possesses no sign changes in expanded form. Divide synthetically with negative possible rational roots until a zero appears in the last entry of the quotient row. Once a zero appears, divide by the same divisor into the quotient row.

-1	1	9	30	46	33	9
		-1	-8	-22	-24	-9
-1	1	8	22	24	9	0
		-1	-7	-15	-9	
	1	7	15	9	0	

Since the same divisor divided evenly into two different factors of the polynomial, the polynomial has multiplicity of roots. If the same divisor had not yielded a zero, then that root would not have multiplicity, but some other root may. To show multiplicity, divide until a root divides evenly into two different factors of the polynomial.

Example 3
Finding Multiplicity

Consider $d(t) = t^5 + 9t^4 + 30x^3 + 46x^2 + 33x + 9$, which has only two roots. Factor $d(t)$.

According to the N Roots Theorem, $d(t)$ must have five roots, not necessarily distinct. If $d(t)$ has only two roots, then all its roots are not distinct and must exhibit multiplicity. One of the two factors must have triple multiplicity while the other must have double multiplicity (because 2 + 3 = 5).

Use Descartes' Rule of Signs to note that the polynomial does not have any positive roots since it possesses no sign changes in expanded form. Divide synthetically with negative possible rational roots until a zero appears in the last entry of the quotient row. Once a zero appears, divide by the same divisor into the quotient row.

$$
\begin{array}{r|rrrrrr}
-1 & 1 & 9 & 30 & 46 & 33 & 9 \\
 & & -1 & -8 & -22 & -24 & -9 \\
\hline
-1 & 1 & 8 & 22 & 24 & 9 & 0 \\
 & & -1 & -7 & -15 & -9 & \\
\hline
-1 & 1 & 7 & 15 & 9 & 0 & \\
 & & -1 & -6 & -9 & & \\
\hline
 & 1 & 6 & 9 & 0 & &
\end{array}
$$

Recognize the three-term quotient with a zero remainder as a quadratic factor, which will yield the remaining root, which must have a multiplicity of two.

$$
\begin{aligned}
&x^2 + 6x + 9 = 0 \\
&(x+3)(x+3) = 0 \\
&x + 3 = 0 \\
&x = -3
\end{aligned}
$$

The roots of $d(t)$ are –1 (triple root) and –3 (double root). Thus, $d(t)$ can be written as a product of its linear factors: $d(t) = (t+1)^3 (t+3)^2$.

Practice Set 2.7

Sketch the graph of the following polynomial functions. Indicate correct end behavior and correct behavior near the *x*-axis.

#1 $g(x) = (x-3)^2(x+2)$

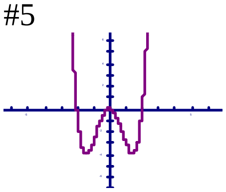
#2 $h(x) = (x+2)(x-3)(x-1)^3$

#3 $f(x) = (x+2)^2(x-1)^3$

#4 $y(x) = x(x-1)(2x+3)^2$

For each of the following sketches, write a polynomial function to the lowest possible degree and with the smallest possible coefficients whose graph would match the sketch. Each unit along the *x*-axis equals one. Each unit along the *y*-axis equals some constant.

#5



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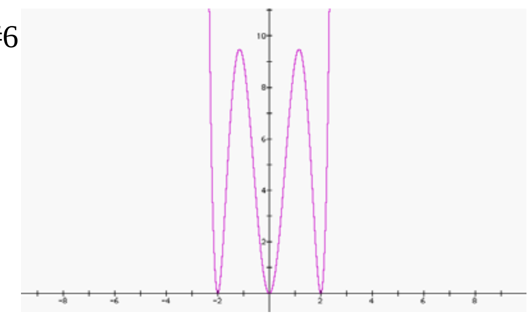

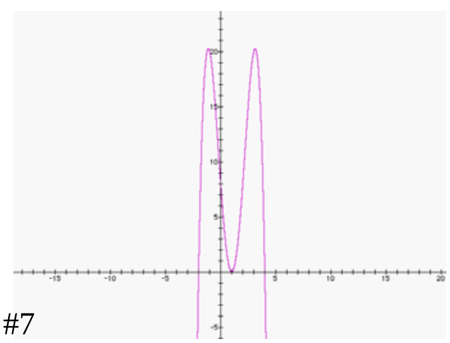

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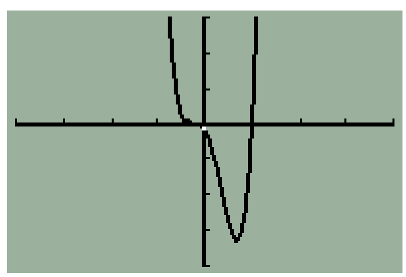
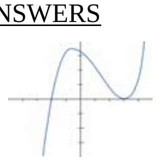


#8

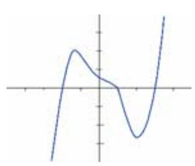
Hint for #8: The leading coefficient is 64, and the negative fractional root is $x = -\frac{1}{4}$ with triple multiplicity.

<u>ANSWERS</u>

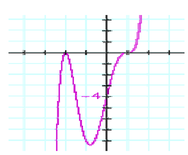
#1



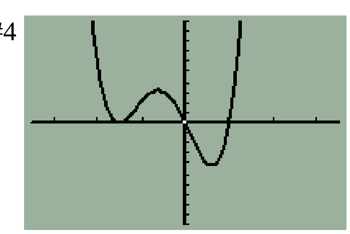
#2



#3



#4



#5 $p(x) = x^2(x+2)(x-2)$
#6 $q(x) = x^2(x+2)^2(x-2)^2$
#7 $r(x) = -1(x+2)(x-4)(x-1)^2$
#8 $T(x) = (x-1)(4x+1)^3$

Assignment 2.7

Problems

Sketch the graph of the following polynomial functions. Indicate correct end behavior and correct behavior near the *x*-axis.

1 $g(x)=(x+1)^2(x-2)^3$

2 $h(x)=x^2(x+7)$

3 $P(x)=(x+5)^2(x-5)^2$

4 $r(x)=x^2+8x+16$