

**Instruction: *Graphing Rational Functions***

Consider  $R(x) = \frac{3x+8}{x+2}$ .

To sketch the graph of  $R(x)$ , one can follow eight steps outlined below:

1. Consider the restrictions on the domain.
2. Find vertical asymptotes.
3. Find horizontal asymptotes.
4. Find slant asymptotes.
5. Find the  $y$ -intercepts.
6. Find the  $x$ -intercepts.
7. Generate a table of values for the function corresponding with different  $x$ -values, especially with  $x$ -values near restrictions on the domain.
8. Graph the asymptotes, intercepts, plot points, and sketch graph.

These eight steps are employed below to graph  $R(x)$ .

First, one must recognize that negative two restricts the domain of the function. The domain includes all real numbers except negative two because substituting negative two for  $x$  yields an undefined value for the function.

Second,  $R(x)$  has a vertical asymptote at  $x = -2$ . Remember vertical asymptotes correspond with the roots of the denominator.

Third,  $R(x)$  has a horizontal asymptote at  $y = 3$ . Since the degree of the numerator equals the degree of the denominator, the function has asymptotic behavior at the quotient of the leading coefficients. The leading coefficient of the numerator is 3. The leading coefficient of the denominator is 1. The quotient yields the asymptotic value:  $3 \div 1 = 3$ . Thus,  $R(x)$  has the horizontal asymptote:  $y = 3$ .

Fourth,  $R(x)$  does not have any slant asymptotes. Oblique asymptotes only occur when the degree of the numerator exceeds the denominator by exactly one.

Fifth, the  $y$ -intercept will occur when  $x = 0$ . Thus,  $R(0)$  yields the  $y$ -intercept:

$$\begin{aligned}R(x) &= \frac{3x+8}{x+2} \\R(0) &= \frac{3(0)+8}{0+2} \\R(0) &= \frac{8}{2} \\R(0) &= 4.\end{aligned}$$

$R(x)$  crosses the  $y$ -axis at  $(0,4)$ .

Lecture 2.13

Sixth, the  $x$ -intercepts occur where the function equals zero. Thus,  $R(x) = 0$  yields the  $x$ -intercepts:

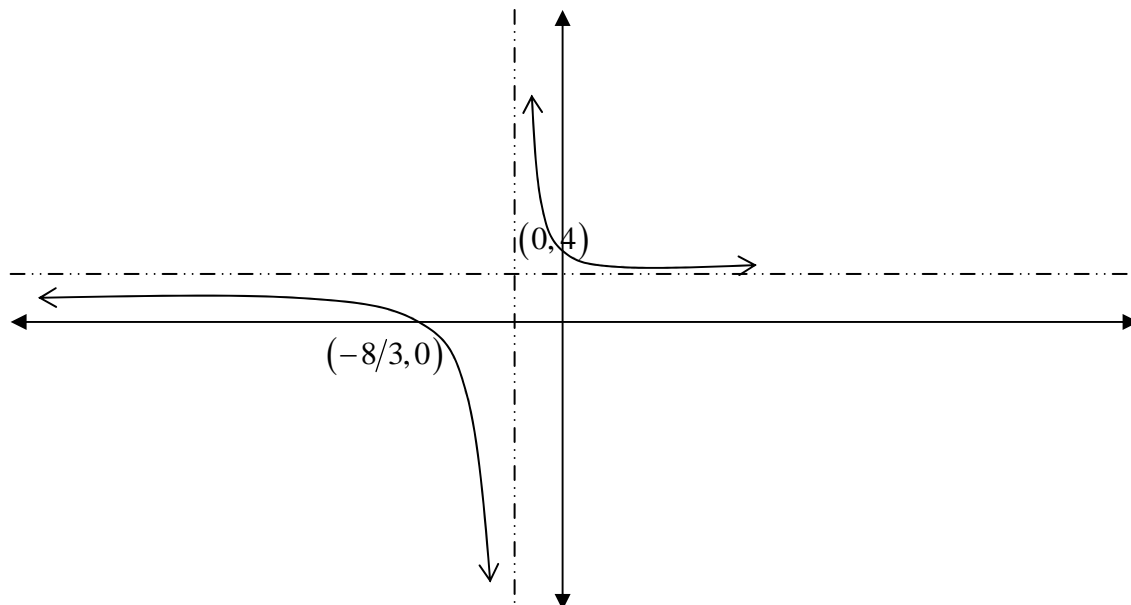
$$\begin{aligned}\frac{3x+8}{x+2} &= 0 \\ (x+2) \cdot \frac{3x+8}{x+2} &= 0 \cdot (x+2) \\ 3x+8 &= 0 \\ 3x &= -8 \\ x &= -\frac{8}{3}.\end{aligned}$$

$R(x)$  crosses the  $x$ -axis at  $(-\frac{8}{3}, 0)$ .

Seventh, evaluating the function for several  $x$ -values generates the following table of values:

$x$	$R(x)$
-10	2.75
-4	2
-3	1
$-\frac{8}{3}$	0
-2.25	-5
-1.75	11
-1	5
0	4
1	$\frac{11}{3}$
2	3.5
10	$\frac{19}{6}$

Eighth, sketching dotted lines for the asymptotes, plotting the intercepts, and plotting the ordered pairs from the table reveals the general behavior of the function from which its graph can be sketched.



**Instruction: *Graphing Rational Functions***

**Example 1**  
**Finding the Intercepts of a Rational Function**

Given  $r(x) = \frac{6-5x}{x+2}$ , find the intercepts of  $r(x)$ .

Substitute zero for  $x$  to find the  $y$ -intercept.

$$r(0) = \frac{6-5(0)}{0+2} = \frac{6}{2} = 3$$

The  $x$ -intercepts of a rational function have  $x$ -values equal to the roots of the numerator provided that these roots are not domain restrictions. Note the domain restrictions of the given function:  $x \neq -2$ . Find the roots of the numerator. If these roots do not equal negative two, they are the  $x$ -values of the  $x$ -intercepts.

$$6 - 5x = 0$$

$$6 = 5x$$

$$\frac{6}{5} = x$$

The  $y$ -intercept is  $(0,3)$ , and the  $x$ -intercept is  $\left(\frac{6}{5}, 0\right)$ .

Example Exercises 2.13

**Example 2**  
**Graphing a Rational Function**

Graph  $q(x) = \frac{-2x}{1-x}$ . Label all intercepts and asymptotes. Show correct asymptotic behavior.

Find the domain restrictions.

$$1 - x \neq 0$$

$$1 \neq x$$

Find the vertical asymptotes. The vertical asymptotes take the form " $x = a$ " where  $a$  is a non-reducible domain restriction.

$$VA: x = 1$$

Find the horizontal asymptote. When the degree of the numerator equals the degree of the denominator, the horizontal asymptote takes the form " $y =$  ratio of leading coefficient of the numerator to the leading coefficient of the denominator."

$$y = -2/-1$$

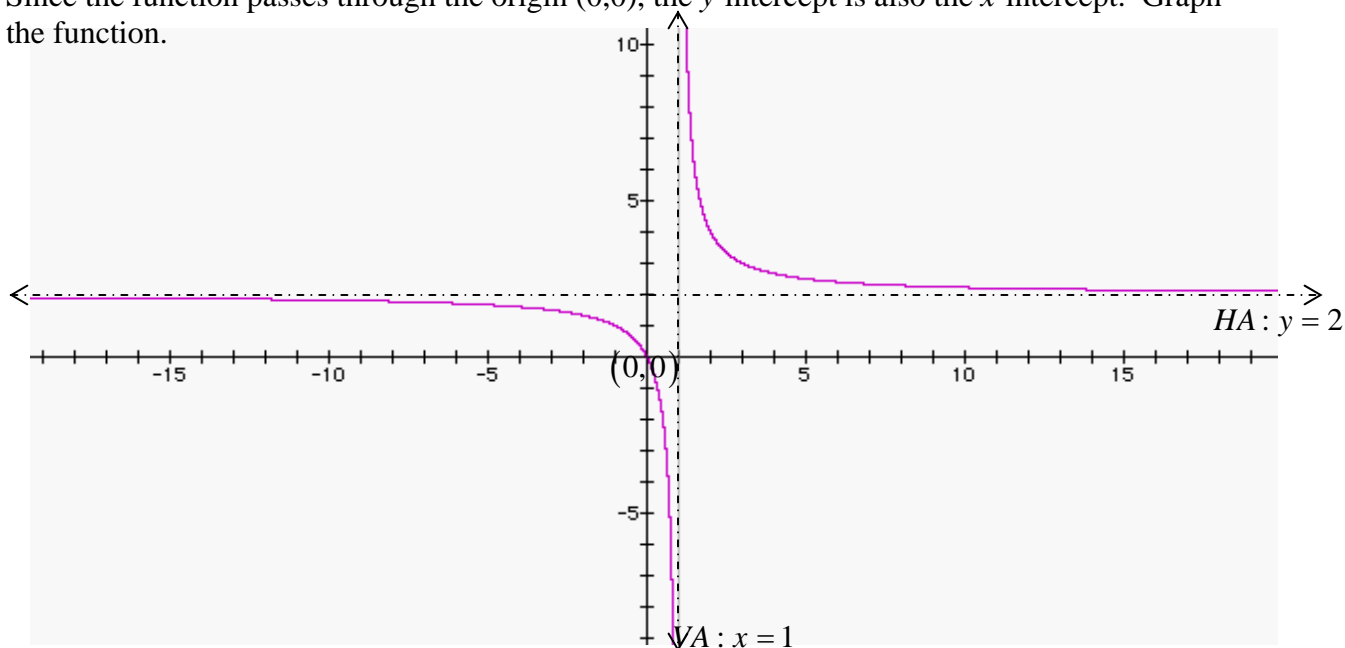
$$HA: y = 2$$

The function does not have a slant asymptote because the degree of the numerator does not exceed the degree of the denominator. Find the  $y$ -intercept.

$$q(0) = \frac{-2(0)}{1-(0)}$$

$$q(0) = 0$$

Since the function passes through the origin  $(0,0)$ , the  $y$ -intercept is also the  $x$ -intercept. Graph the function.



Practice Set 2.13

Sketch the graph the following rational functions. Label the units along each axis.

#1  $R(x) = \frac{2}{x}$

#2  $Q(x) = \frac{3}{x-2}$

#3  $r(x) = \frac{-3}{x+1}$

#4  $q(x) = \frac{x+3}{x-1}$

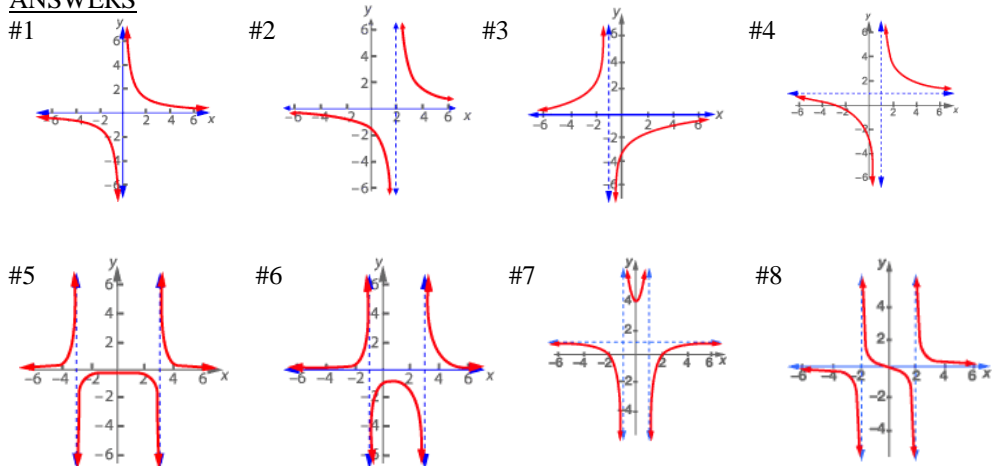
#5  $f(x) = \frac{2}{x^2-9}$

#6  $g(x) = \frac{x+3}{x^2-2x-3}$

#7  $h(x) = \frac{x^2-4}{x^2-1}$

#8  $F(x) = \frac{x}{x^2-4}$

ANSWERS



## Assignment 2.13

### Problems

Neatly sketch the graph the following rational functions. Draw the graphs with fastidious care. Label all asymptotes. Label all intercepts. If you do not use graph paper, use a ruler to evenly distribute the units along each axis.

1  $f(x) = \frac{2x+6}{x+1}$

2  $g(x) = \frac{x^2-4}{x}$

3  $h(x) = \frac{3-x}{x}$

4  $r(x) = \frac{x}{x^2-x-20}$

**Instruction: Graphing Reducible Rational Functions**

$$R(x) = \frac{x+1}{x^2-1}$$

The previous lecture provided a 8-step process for graphing rational functions. This lecture will use the same 8-step process, but it will also consider rational functions that contain reducible rational functions like  $R(x)$  above.

$R(x)$  has two restrictions on its domain:  $x = \pm 1$ , but it has only one vertical asymptote:

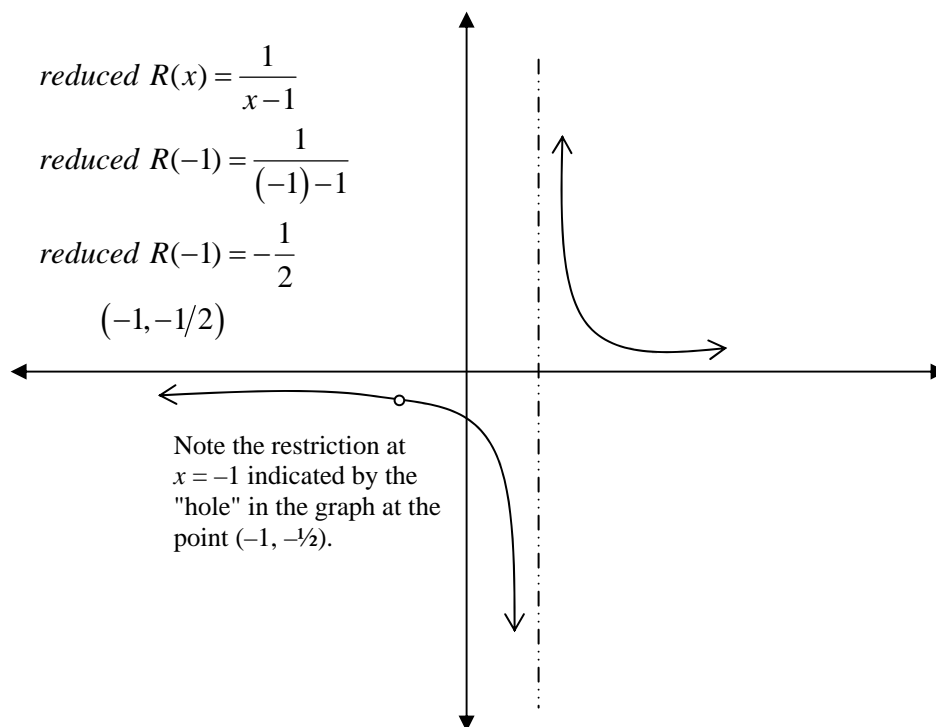
$$R(x) = \frac{x+1}{x^2-1}$$

$$R(x) = \frac{x+1}{(x+1)(x-1)}$$

$$R(x) = \frac{x+1}{(x+1)(x-1)}$$

$$R(x) = \frac{1}{x-1}, x \neq -1$$

Thus,  $R(x)$  has a vertical asymptote at  $x = 1$ . A tricky problem occurs while graphing rational functions like these. The graph does not exist at  $x = -1$ . Normally, the vertical asymptotes remind the student of restrictions in the domain, but this graph does not have a vertical asymptote at  $-1$ . Consequently, students must plot a "hole" so to speak to represent the domain restriction. To find where the "hole" occurs (the limit of the function as  $x$  approaches  $-1$ ), substitute the domain restriction into the reduced function:



Example Exercises 2.14

**Instruction: Graphing Reducible Rational Functions**

**Example 1  
Finding Points of Discontinuity**

Given  $r(x) = \frac{x^2 - 25}{x^2 - 3x - 10}$ . Find the vertical asymptote and the point approached by the graph at  $x = 5$ .

Find the domain restrictions by finding the roots of the denominator.

$$\begin{aligned}x^2 - 3x - 10 &= 0 \\(x - 5)(x + 2) &= 0 \\x - 5 = 0, \quad x + 2 = 0 \\x = 5, \quad x = -2 \\ \therefore \text{ for } r(x) \quad x \neq 5 \text{ and } x \neq -2\end{aligned}$$

Reduce the rational function.

$$r(x) = \frac{x^2 - 25}{x^2 - 3x - 10} = \frac{(x+5)\cancel{(x-5)}}{\cancel{(x-5)}(x+2)} = \frac{x+5}{x+2}, x \neq 5$$

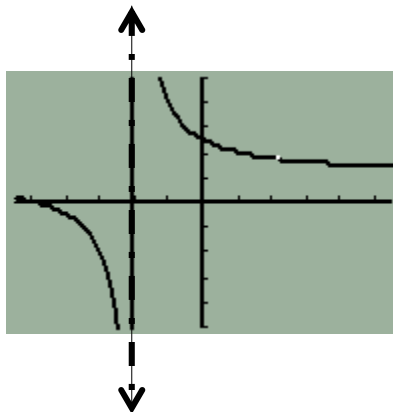
The vertical asymptote takes the form " $x = a$ " where  $a$  is the root(s) of the denominator after reduction. Identify the vertical asymptote.

$$VA: x = -2$$

Find the point of discontinuity. The domain restriction unrepresented by an asymptote corresponds to a point of discontinuity. As  $x$ -approaches the domain restriction of five, the function will approach the value that the reduced form of the function takes when  $x$  equals five.

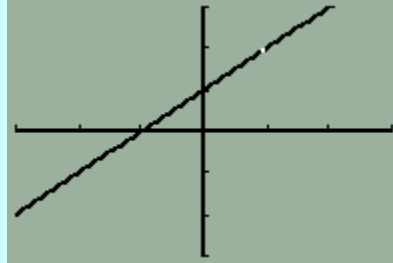
$$\frac{5+5}{5+2} = \frac{10}{7}$$

As  $x$ -approaches five, the function will approach  $(5, 10/7)$ . On the graph below, the vertical dotted line represents the vertical asymptote while the white dot represents the point of discontinuity.



**Example 2**  
**Graphing Reducible Rational Functions**

Consider the graph below.



Write a rational function whose graph matches the above sketch assuming the white dot represents the function's only domain restriction.

Note that the graph is linear with a domain restriction at  $x = 1$ . The line passes through  $(0, 1)$  and has a positive slope that rises vertically one unit for every unit increase horizontally, so if the line takes the form  $y = mx + b$ , then  $m = 1$  and  $b = 1$ . Consequently, the function should have a domain restriction at  $x = 1$  and should reduce to  $y = x + 1$ .

$$f(x) = \frac{(x+1)(x-1)}{x-1}$$

$$f(x) = \frac{x^2 - 1}{x - 1}$$

Practice Set 2.14

Graph the following rational functions.

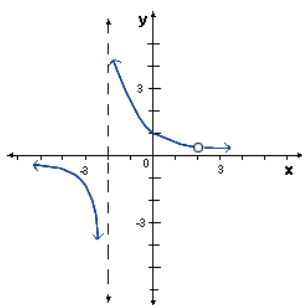
#1  $Q(x) = \frac{x-2}{x^2-4}$

#2  $q(x) = \frac{x^2-9}{x-3}$

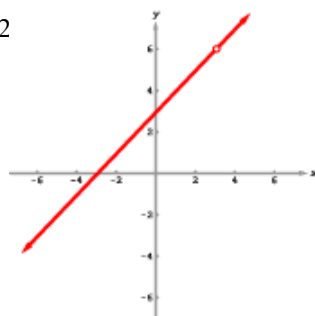
#3  $r(x) = \frac{x+3}{9-x^2}$

ANSWERS

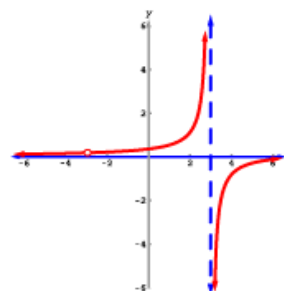
#1



#2



#3



Assignment 2.14

**Problems**

Neatly sketch the graph of the following rational functions. Label all asymptotes. Label all intercepts. If you do not use graph paper, use a ruler to distribute the units along each axis evenly. Sketch the graphs with fastidious care.

1  $f(x) = \frac{x^2 - 4}{x + 2}$

2  $g(x) = \frac{2x + 6}{x^2 - x - 12}$