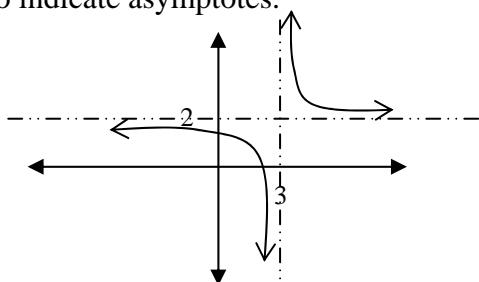


**Instruction: *Horizontal Asymptotes of Rational Functions***

When a graph approaches a line, that line is an asymptote. The function below has a vertical asymptote of  $x = 3$ , and a horizontal asymptote at  $y = 2$ . Since asymptotes are not part of the graph, dotted lines are used to indicate asymptotes.



Many rational functions have asymptotes. Finding the asymptotes is an important step in graphing a rational function, and this lecture will discuss how to find the horizontal asymptotes of a rational function.

A horizontal asymptote is a line that the graph will approach as  $x$ -values approach infinity and negative infinity. Horizontal asymptotes are linear equations of the form  $y = b$ , where  $b$  is a real number equal to the  $y$ -value of the  $y$ -intercept of the asymptote.

Consider  $r_1(x) = \frac{5x+1}{x^2-4}$ . Since the degree of the denominator (two) exceeds the degree of the numerator (one), the function has a horizontal asymptote at  $y = 0$ . All rational functions whose denominator's degree exceeds the numerator's degree have  $y = 0$  as a horizontal asymptote.

If the degree of the denominator equals the degree of the numerator, the function will approach  $y =$  the quotient of the leading coefficients of the numerator and the denominator. For example, consider  $r_2(x)$  below.

$$r_2(x) = \frac{6x^2 + 7}{2x^2 + 9}$$

The leading coefficient of the numerator is six. The leading coefficient of the denominator is 2. The quotient of six and two is three; thus,  $r_2(x)$  has a horizontal asymptote at  $y = 3$ .

If the degree of the numerator exceeds the degree of the denominator, the function will not have a horizontal asymptote.

**Instruction: *Horizontal Asymptotes***

**Example 1  
Finding the Horizontal Asymptote of a Rational Function**

Consider  $R(x) = \frac{N(x)}{D(x)}$  where  $N(x)$  and  $D(x)$  are greater than zero-degree polynomial functions. Is the following statement true or false? Statement:  $R(x)$  must have a horizontal asymptote.

Not all rational functions have horizontal asymptotes. If  $R(x) = N(x)/D(x)$ , then  $R(x)$  will not have an asymptote if the degree of  $N(x)$  is greater than the degree of  $D(x)$ . The statement is false.

**Example 2  
Finding the Horizontal Asymptote of a Rational Function**

Consider  $Q(x) = \frac{17 - 2x^2}{3x^2 - 5}$ . Identify the horizontal asymptote of  $Q(x)$ .

For any rational function of the form  $R(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{c_p x^p + c_{p-1} x^{p-1} + \cdots + c_1 x + c_0}$  where  $n = p$ ,

$R(x)$  will approach the horizontal line  $y = a_n/c_p$ . For  $Q(x)$ ,  $n = 2 = p$ , i.e., the degree of the numerator equals the degree of the denominator. Moreover,  $a_n = -2$  and  $c_p = 3$ , i.e., the leading coefficient of the numerator is negative two and the leading coefficient of the denominator is three. Consequently, the horizontal asymptote of  $Q(x)$  is  $y = -2/3$ .

**Example 3  
Finding the Horizontal Asymptote of a Rational Function**

Consider  $V(x) = \frac{200}{(x+3)^2}$ . Identify the horizontal asymptote of  $V(x)$ .

For any rational function of the form  $R(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{c_p x^p + c_{p-1} x^{p-1} + \cdots + c_1 x + c_0}$  where  $n < p$ ,

$R(x)$  will approach the horizontal line  $y = 0$ . For  $V(x)$ ,  $n < p$ , i.e., the degree of the numerator (0) is less than the degree of the denominator (2). Consequently, the horizontal asymptote of  $V(x)$  is  $y = 0$ .

## Practice Set 2.11

Find the horizontal asymptotes of the following rational functions. Some of the functions may not have a horizontal asymptote.

$$\#1 \quad f(x) = \frac{7x^2 + 3}{x^2 + 11x + 28}$$

$$\#2 \quad r(x) = \frac{x + 2}{x^2 + 4x + 3}$$

$$\#3 \quad R(x) = \frac{x^2 - 4}{6x^2 - 7x - 5}$$

$$\#4 \quad q(x) = \frac{8}{x^2 - 4}$$

$$\#5 \quad Q(x) = \frac{x^2 - 4}{x + 2}$$

$$\#6 \quad d(x) = \frac{-x^3 + x}{x^3 + 2x^2 - 7x - 2}$$

$$\#7 \quad D(x) = \frac{3x^3 + 2x^2 - 7x - 5}{4x^3 + 1}$$

$$\#8 \quad g(x) = \frac{x^3 - 8}{x - 1}$$

$$\#9 \quad h(x) = \frac{12}{x^3 + 4x^2 - x - 4}$$

$$\#10 \quad p(x) = \frac{x - 1}{x - 1}$$

### ANSWERS

#1 HA:  $y = 7$

#2 HA:  $y = 0$

#3 HA:  $y = \frac{1}{6}$

#4 HA:  $y = 0$

#5 HA: None, the rational function does not have a horizontal asymptote.

#6 HA:  $y = -1$

#7 HA:  $y = \frac{3}{4}$

#8 HA: None, the rational function does not have a horizontal asymptote.

#9 HA:  $y = 0$

#10 HA: None, the rational function does not have a horizontal asymptote.

## Assignment 2.11

### Problems

Find the horizontal asymptotes of the following rational functions. Some of the functions may not have a horizontal asymptote.

1  $f(x) = \frac{x}{x^2 - 9}$

2  $g(x) = \frac{x+3}{x^2 - x - 30}$

3  $h(x) = \frac{x^2 - 4}{x^2 + 9x + 14}$

4  $j(x) = \frac{17 - 3x}{2x + 1}$

5  $R(x) = \frac{x^3 - 8}{x + 2}$

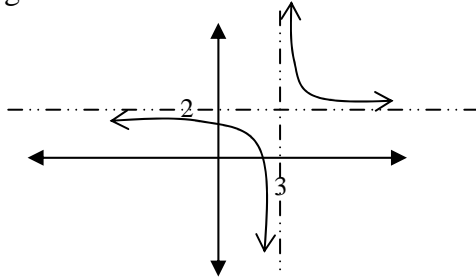
6  $r(x) = \frac{1}{x}$

7  $Q(x) = \frac{x^2 + 2x + 7}{x + 1}$

8  $q(x) = \frac{6 - 5x^2}{x^2}$

**Instruction: *Slant (or Oblique) Linear Asymptotes of Rational Functions***

When a graph approaches a line, that line is an asymptote. The function below has a vertical asymptote of  $x = 3$ , and a horizontal asymptote at  $y = 2$ . Since asymptotes are not part of the graph, they are indicated using dotted lines.



The above graph shows two asymptotes, one vertical and one horizontal. Some rational functions have oblique linear asymptotes also called slant asymptotes.

An oblique asymptote is a line that the function will approach as the  $x$ -values approach infinity or negative infinity. Oblique asymptotes can be described using an equation of the form  $y = mx + b$  where  $m$  and  $b$  are real numbers and  $m \neq 0$ .

Oblique asymptotes occur in rational functions that have a numerator with a degree that is exactly one larger than the degree of the denominator. For example consider  $R(x)$  below.

$$R(x) = \frac{x^3 - 7x - 6}{x^2 - 4}$$

Since the degree of the numerator (three) exceeds the degree of the denominator (two) by exactly one, the function has a slant asymptote. The slant asymptote will take the equation:  $y =$  the quotient without the remainder (if there is no remainder, there is no linear asymptote). Consequently, performing long division finds the equation of the slant asymptote:

$$\begin{array}{r} x \\ x^2 + 0x - 4 \overline{) x^3 + 0x^2 - 7x - 6} \\ \underline{-(x^3 + 0x^2 - 4x)} \phantom{-6} \\ -3x - 6 \end{array}$$

Discard the remainder  $(-3x - 6)$ . The quotient without the remainder is the slant asymptote. So,  $R(x)$  has a slant asymptote at  $y = x$ . Even though the remainder is discarded while determining the equation of the slant asymptote, it is sometimes important. For our class, the remainder is only important because it shows that the denominator does not cancel completely with factors in the numerator. If this did happen (i.e., if there was no remainder), there would not be a linear slant asymptote.

**Instruction: *Slant (Oblique) Asymptotes***

**Example 1  
Finding the Slant Asymptote of a Rational Function**

Consider  $R(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0}{c_p x^p + c_{p-1} x^{p-1} + \dots + c_1 x + c_0}$  where  $D(x) \neq 0$ ,  $p \neq 0$ , and  $n = p + 1$ .

Under what conditions is the following statement true? Statement:  $R(x)$  will approach a linear slant asymptote as  $x$ -values approach infinity and negative infinity.

If  $R(x) = N(x)/D(x)$  and the degree of  $N(x)$  is exactly one greater than the degree of  $D(x)$ , which is greater than zero, then  $R(x)$  will approach a slant asymptote as  $x$ -values approach infinity and negative infinity provided that  $D(x)$  is not a factor of  $N(x)$ . The statement is true under the condition that  $D(x)$  is not a factor of  $N(x)$ . In other words, the statement is true if the quotient  $N(x)/D(x)$  has a non-zero remainder.

**Example 2  
Finding the Slant Asymptote of a Rational Function**

Consider  $Q(x) = \frac{5 - 6x^3}{3x^2 + 1}$ . Identify the slant asymptote of  $Q(x)$ .

Examine the degree of the numerator,  $n$ , and the degree of the denominator,  $p$ :  $3 = 2 + 1$ .

Since the degree of the numerator exceeds the degree of the denominator by exactly one, the rational function may have a slant asymptote provided that the denominator is not a factor of the numerator. Divide the denominator into the numerator.

$$\begin{array}{r} \phantom{3x^2 + 0x + 1} \overline{) -6x^3 + 0x^2 + 0x + 5} \\ \underline{-(-6x^3 + 0x^2 - 2x)} \phantom{+ 5} \\ 2x + 5 \end{array}$$

∴

$$\frac{5 - 6x^3}{3x^2 + 1} = -2x + \frac{2x + 5}{3x^2 + 1}$$

Since the quotient has a non-zero remainder, the denominator is not a factor of the numerator, and the slant asymptote takes the form "y = quotient without remainder." Consequently, the slant asymptote is  $y = -2x$ .



Practice Set 2.12

Find the oblique asymptotes (also called slant asymptotes) of the following rational functions. Some of the functions may not have oblique linear asymptotes.

#1  $f(x) = \frac{x^2 + 1}{x - 2}$

#2  $r(x) = \frac{x - x^2}{x + 2}$

#3  $R(x) = \frac{2x^2 + 3}{4 - x}$

#4  $q(x) = \frac{x^3 - 4x^2 + x + 6}{x^2 + x - 2}$

#5  $Q(x) = \frac{x^2 - 36}{x - 6}$

#6  $d(x) = \frac{x^3 - x}{x^4 - 16}$

#7  $D(x) = \frac{x^3 + 3x^2 + 2x + 6}{x^2 + 2}$

#8  $g(x) = \frac{x^3 + 4x^2 + 5x + 2}{x^2 + x + 1}$

#9  $h(x) = \frac{10x^2 - 3}{x + 7}$

#10  $p(x) = \frac{x^2 - 1}{x + 2}$

ANSWERS

#1 SA:  $y = x + 2$

#2 SA:  $y = -x + 3$

#3 SA:  $y = -2x - 8$

#4 SA:  $y = x - 5$

#5 SA: None, the rational function does not have an oblique asymptote.

#6 SA: None, the rational function does not have a oblique asymptote.

#7 SA: None, the rational function does not have a oblique asymptote.

#8 SA:  $y = x + 3$

#9 SA:  $y = 10x - 70$

#10 SA:  $y = x - 2$

Assignment 2.12

**Problems**

Find the slant asymptotes of the following rational functions. Some of the functions may not have slant asymptotes.

1  $f(x) = \frac{x^2 + 9x + 20}{x + 7}$

2  $g(x) = \frac{x^3 - 8}{x + 2}$

3  $h(x) = \frac{x^2 - 10x + 9}{x + 8}$

4  $R(x) = \frac{x^2 - x - 12}{x - 4}$