College Algebra

Instruction: ***Bisection Method of Estimating Roots with the Root Location Theorem***

The Root Location Theorem, stated below, tells us that a polynomial function must have a real root between two *x*-values whose corresponding *y*-coordinates have opposite signs.

> **Root Location Theorem:** *If* P(x) *is a polynomial function defined in such a way that for real numbers* a *and* b*,* P(a) *and* P(b) *differ in sign, then there exists at least one real root between* a *and* b.

Consider $f(x) = x^3 + 9x^2 - 5x - 45$ and the synthetic division algorithm performed below.

2	1	9	–5	–45
		2	22	34
	1	11	17	–11

Using 2 as the divisor, the division algorithm yields –11 as the remainder. The Remainder Theorem concludes that when $x = 2$ the function will equal –11; therefore, $f(x)$ passes through the ordered pair (2, –11).

3	1	9	–5	–45
		3	36	93
	1	12	31	48

Using 3 as the divisor, the division algorithm yields 48 as the remainder. Thus, $f(x)$ passes through the coordinate (3,48).

Now consider the position of the two ordered pairs (2,–11) and (3,48). Graph these two coordinates on a Cartesian plane (the *x*-*y* plane), and notice that one falls below the *x*-axis while the other lies above the *x*-axis. The Root Location Theorem concludes that at least one root must exist between 2 and 3 on the *x*-axis since the graph must cross the *x*-axis somewhere between –11 and 48. Logic suggests that if there is only one root it will probably be closer to 2 than to 3 since (2, –11) is closer to the *x*-axis.

To estimate the value of the root to the nearest tenth, perform the division algorithm with increasing values between 2 and 3. For instance, divide by 2.1, then 2.2, then 2.3, etc., watching for the remainder to change signs:

2.2	1	9	–5	–45
		2.2	24.64	43.208
	1	11.2	19.64	–1.792

2.3	1	9	–5	–45
		2.3	25.99	48.277
	1	11.3	20.99	3.277

A root must exist between 2.2 and 2.3 since the remainder changed signs between these two divisors. Since the remainder with 2.2 is closer to the *x*-axis than the remainder with 2.3, the root estimated to the nearest tenth is 2.2.

Instruction: *Root Location Theorem and Bisection Method*

Example 1
Root Location Theorem

Consider $P(x) = x^4 + x^3 - x^2 - 3x - 6$. Show that $P(x)$ has at least one root between 1 and 2.

Divide synthetically by one.

1	1	1	−1	−3	−6
		1	2	1	−2
	1	2	1	−2	−8

The Remainder Theorem indicates that $P(1) = -8$. Divide synthetically by two.

2	1	1	−1	−3	−6
		2	6	10	14
	1	3	5	7	8

The Remainder Theorem indicates that $P(2) = 8$. Since $P(1)$ differs in sign from $P(2)$, a root exists between 1 and 2 by the Root Location Theorem.

Example 2
Bisection Method

Consider $y(x) = x^4 + 2x^3 - x - 6$. Estimate the positive root of $y(x)$ to the nearest tenth.

Descartes' Rule of Signs indicates that $y(x)$ has a positive root because it has one sign change in expanded form and descending order. Note that $y(0) < 0$. Perform synthetic division with integers until a positive remainder occurs.

$$\begin{array}{r|rrrrr} 1 & 1 & 2 & 0 & -1 & -6 \\ & & 1 & 3 & 3 & 2 \\ \hline & 1 & 3 & 3 & 2 & -4 \end{array} \qquad \begin{array}{r|rrrrr} 2 & 1 & 2 & 0 & -1 & -6 \\ & & 2 & 8 & 16 & 30 \\ \hline & 1 & 4 & 8 & 15 & 24 \end{array}$$

According to the Root Location Theorem, a root occurs between 1 and 2 because $P(1)$ differs in sign from $P(2)$. Divide synthetically by 1.1, 1.2, etc., until a positive remainder occurs.

$$\begin{array}{r|rrrrr} 1.1 & 1 & 2 & 0 & -1 & -6 \\ & & 1.1 & 3.41 & 3.751 & 3.0261 \\ \hline & 1 & 3.1 & 3.41 & 2.751 & -2.9739 \end{array}$$

$$\begin{array}{r|rrrrr} 1.2 & 1 & 2 & 0 & -1 & -6 \\ & & 1.2 & 3.84 & 4.608 & 4.3296 \\ \hline & 1 & 3.2 & 3.84 & 3.608 & -1.6704 \end{array}$$

$$\begin{array}{r|rrrrr} 1.3 & 1 & 2 & 0 & -1 & -6 \\ & & 1.3 & 4.29 & 5.577 & 5.9501 \\ \hline & 1 & 3.3 & 4.29 & 4.577 & -0.0499 \end{array}$$

$$\begin{array}{r|rrrrr} 1.4 & 1 & 2 & 0 & -1 & -6 \\ & & 1.4 & 4.76 & 6.664 & 7.9296 \\ \hline & 1 & 3.4 & 4.76 & 5.664 & 1.9296 \end{array}$$

Since $P(1.3)$ differs in sign from $P(1.4)$, a root exists between 1.3 and 1.4 by the Root Location Theorem. Since $P(1.3)$ is closer to zero than $P(1.4)$, 1.3 is the value of the root rounded to the nearest tenth.

Practice Set 2.9

#1 Suppose that a polynomial function P(x) is defined in such a way that $P(5) = -3$ and $P(6) = 4$. What conclusion does the Root Location Theorem allow you to make?

#2 The following polynomial has an irrational root between 2 and 3. Estimate the value of the root to the nearest hundredth using the Root Location Theorem and the Remainder Theorem.

$$f(x) = x^3 - 2x^2 - x + 1$$

#3 The following polynomial has an irrational root between 1.5 and 2. Estimate the value of the root to the nearest hundredth using the Root Location Theorem and the Remainder Theorem.

$$p(x) = 2x^4 - 4x^2 + 3x - 6$$

#4 The following polynomial has an irrational root between 2 and 3. Estimate the value of the root to the nearest hundredth using the Root Location Theorem and the Remainder Theorem.

$$n(x) = 2x^3 - 8x^2 + x + 16$$

ANSWERS
#1 The polynomial $P(x)$ has a root between 5 and 6.
#2 $x \approx 2.25$
#3 $x \approx 1.52$
#4 $x \approx 2.39$

Practice Set 2.9

<u>ANSWERS</u>

#1 The polynomial $P(x)$ has a root between 5 and 6.
#2 $x \approx 2.25$
#3 $x \approx 1.52$
#4 $x \approx 2.39$

Study Exercise 2.9

Problems

The following polynomial has an irrational root between 2 and 3. Use the bisection method to estimate the value of the root to the nearest tenth.

#1 $g(x) = x^3 + x^2 - 6x - 6$