Instruction: Expected Value

The start of this lecture discusses *expected value* or mathematical expectation. Expected value is a special sum associated with the probability distribution of a game. A *game*, defined below, is an experiment.

> A *game* is an experiment with a defined set of discrete values called payoffs for a random variable, X, that represents a set of mutually exclusive events whose union comprises the sample space of the experiment.

The probability distribution associated with a game lists the values of a random variable, X, and the probabilities, $P(X)$, corresponding to the values of the random variable. The values of the random variable will be "payoffs" measured in dollars, time units, spaces on a board, or other units of measures. The *expected value* associated with a probability distribution equals the sum of the products of the values of the payoffs of the random variable and the probabilities corresponding to those payoffs. In other words, the mean μ of a probability distribution is the *expected value* of its random variable.

> If X is a discrete random variable with payoffs $X_1, X_2, \ldots, X_n$, occurring with probabilities $P(X_1), P(X_2), \ldots, P(X_n)$, respectively, then the *expected value*, denoted $E(X)$, is given by the sum of the products:
>
> $$E(X) = \mu = \sum_{i=1}^{n} X_i P(X_i) = X_1 \cdot P(X_1) + X_2 \cdot P(X_2) + \cdots + X_n \cdot P(X_n)$$

Consider a roulette wheel with thirty-eight equally-likely possible outcomes. Typically, a bet placed on a single number pays thirty-five to one, meaning a winning one-dollar bet earns thirty-five dollars. The table below represents a probability distribution for a one-dollar bet placed on one spin of the roulette wheel.

X	–\$1.00	\$35.00
$P(X)$	$\frac{37}{38}$	$\frac{1}{38}$

The sum of products below gives $E(X)$, the expected value of this experiment.

$$E(X) = -1 \cdot \frac{37}{38} + 35 \cdot \frac{1}{38}$$

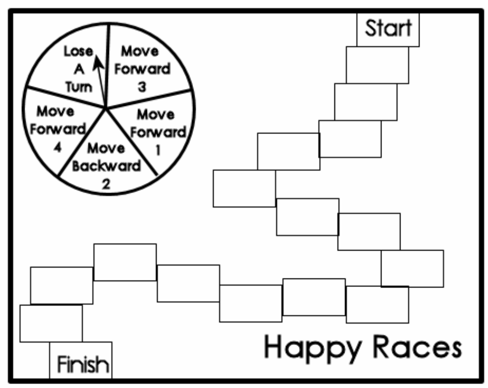
$$E(X) = -\frac{37}{38} + \frac{35}{38}$$

$$E(X) = -\frac{2}{38} \approx -0.053$$

Thus, the expected value of the one dollar bet on a single number and single spin of a roullette wheel equals a loss of about five cents.

Significantly, the expected value does not equal the amount the player might win or lose. Accordingly, expected value has nothing to do with the subjective expectations of the player. The player may optimistically expect to win thirty-five dollars, or the player may realistically expect to lose one dollar. No rational player would expect to lose five cents. Indeed, it is impossible to lose five cents on a single spin. The term "expected" comes from the average payoff per game after a large number of trials. The roullette player can expect to lose an average of five cents per game if he/she plays the game a large number of times.

Consider the board game with the unbiased spinner in Figure 1. Assume each player



spins the dial on his or her turn and moves along the board according to the result of the spin. Any adult who plays a game like this may have found him/herself wondering if the game will ever end. In other words, "Can a player reasonably expect to reach the finish space?" The question involves expected value. The expected value associated with a player's turn is calculated below.

$$E(X) = 3\left(\frac{1}{5}\right) + 1\left(\frac{1}{5}\right) - 2\left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) + 0\left(\frac{1}{5}\right)$$

$$E(X) = 3(0.2) + 1(0.2) - 2(0.2) + 4(0.2) + 0(0.2)$$

$$E(X) = 0.6 + 0.2 - 0.4 + 0.8 = 1.2$$

According to the result, a player can expect to average a gain of 1.2 spaces after a large number of turns, so, yes, the player can expect to finish the game.

Another question associated with expected value involves fairness. Previously, we have equated fairness with a lack of bias. If all the outcomes of a die are equally likely the die is said to be fair. With games, however, fairness equates to a zero expected value. Consider a game that pays according to the results of the unbiased spinner below.

The expected value, calculated to be zero below, shows that this game is *fair*.

$$E(X) = \$8(0.25) + \$2(0.25) - \$5(0.5)$$
$$E(X) = \$2 + \$0.5 - \$2.5$$
$$E(X) = \$0$$

Instruction: Standard Deviation of a Discrete Random Variable

The variance of a probability distribution is calculated by multiplying each possible squared deviation $\left[X_i - E(X)\right]^2$ by its corresponding probability $P(X_i)$ and then finding the sum of the resulting products. Accordingly, the equation below defines σ^2, the variance of a discrete random variable.

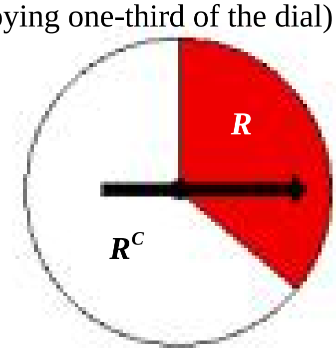
> The *variance of a discrete random variable*, denoted σ^2, is given by
> $$\sigma^2 = \sum_{i=1}^{n}\left(\left[X_i - E(X)\right]^2 \cdot P(X_i)\right)$$
> where X_i equals the *i*th outcome of the discrete random variable, and $P(X_i)$ equals the probability of occurrence of the *i*th outcome of the discrete random variable.

As with the population standard deviation, the *standard deviation of a discrete random variable* equals the square root of its variance.

> The *standard deviation of a discrete random variable*, denoted σ, is given by
> $$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_{i=1}^{n}\left(\left[X_i - E(X)\right]^2 \cdot P(X_i)\right)}$$
> where X_i equals the *i*th outcome of the discrete random variable, and $P(X_i)$ equals the probability of occurrence of the *i*th outcome of the discrete random variable.

Instruction: The Binomial Probability Distribution

Imagine a game that requires spinning the dial below. Imagine a player that "wins" if the arrow lands on the *R*-sector (occupying one-third of the dial) only once in the two spins.



A natural question asks, "What is the probability that the player wins?" To calculate the probability that the player "wins," we will think of each spin as a trial of an experiment, and we will consider two events, R and R^C. Since R occupies one-third of the dial, $P(R) = 1/3$ and $P(R^C) = 2/3$.

Recall that the player "wins" only if the arrow lands on the R-sector exactly once in two spins. In other words, the player could win if R occurs on the first spin *and* R does not occur on the second spin, <u>*or*</u> if R does not occur on the first spin *and* R does occur on the second spin. Let R_1 represent the event that R occurs on the first spin, R_1^C represent the event that R^C occurs on the first spin, R_2 represent the event that R occurs on the second spin, and R_2^C represent the event that R^C occurs on the second spin. Applying the *Addition Rule of Probability for Mutually Exclusive Events* together with the *Multiplication Rule of Probability of Independent Events* we can find $P(\text{winning})$ as below.

$$P(\text{winning}) = P(R_1)\cdot P(R_2^C) + P(R_2)\cdot P(R_1^C)$$
$$P(\text{winning}) = \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{3}$$
$$P(\text{winning}) = \frac{4}{9}$$

This probability can be written as below.

$$P(\text{winning}) = C(2,1)\cdot\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^{2-1}$$

This is a specific example of the *Binomial Probability Formula* stated below in general terms.

Let $\mathcal{W}$ represent exactly X number of occurrences of success in n trials. If the probability of success remains constant throughout n trials, the probability of $\mathcal{W}$ is given by

$$P(\mathcal{W}) = C(n,X)\cdot[p]^X\cdot[1-p]^{n-X}$$

where p equals the probability of success.

Applying the formula, to the situation above, we note that $n = 2$ because there were two spins, $X = 1$ because the player won only if R occurred exactly once, $p = 1/3$. Thus,

$$P(\mathcal{W}) = C(n,x)\cdot[p]^X\cdot[1-p]^{n-X}$$
$$P(\mathcal{W}) = C(2,1)\cdot\left(\frac{1}{3}\right)^1\cdot\left(\frac{2}{3}\right)^{2-1}$$
$$P(\mathcal{W}) = 2\cdot\frac{1}{3}\cdot\frac{2}{3}$$
$$P(\mathcal{W}) = \frac{4}{9}$$

Let's consider an example from meteorology. Consider a set of weather conditions such that a meteorologist can determine from past data that there exists a 30% chance for rain (a 0.3 probability for rain). What is the probability, that it will rain at most three times in ten instances of these weather conditions? To answer this question, we first note the phrase "at most" and recall that the *Binomial Probability Formula* applies to an exact number of occurrences. Nevertheless, the *Binomial Probability Formula* will help us arrive at an answer to the question. Let *R* represent the event that it rains under these conditions, and Let $\mathcal{W}_1$ represent exactly one instance of rain in ten instances of the given set of conditions. $P(\mathcal{W}_1)$, then, represents the probability that there is one occurrence of *R* in ten trials:

$$
\begin{aligned}
P(\mathcal{W}_1) &= C(n,X)\cdot\left[P(R)\right]^X\cdot\left[1-P(R)\right]^{n-X}\\
P(\mathcal{W}_1) &= C(n,X)\cdot\left[P(R)\right]^X\cdot\left[P(R^C)\right]^{n-X}\\
P(\mathcal{W}_1) &= C(10,1)\cdot(0.3)^1\cdot(0.7)^{10-1}\\
P(\mathcal{W}_1) &= 10\cdot(0.3)\cdot(0.7)^9\\
P(\mathcal{W}_1) &\approx 0.1210608
\end{aligned}
$$

Now, let $\mathcal{W}_2$ represent exactly two instances of rain and $\mathcal{W}_3$ represent exactly three instances of rain, and find $P(\mathcal{W}_2)$ and $P(\mathcal{W}_3)$ in the same manner.

$$
\begin{aligned}
P(\mathcal{W}_2) &= C(n,X)\cdot\left[P(R)\right]^X\cdot\left[P(R^C)\right]^{n-X}\\
P(\mathcal{W}_2) &= C(10,2)\cdot(0.3)^2\cdot(0.7)^{10-2}\\
P(\mathcal{W}_2) &= 45\cdot(0.09)\cdot(0.7)^8\\
P(\mathcal{W}_2) &\approx 0.2334744
\end{aligned}
$$

$$
\begin{aligned}
P(\mathcal{W}_3) &= C(n,X)\cdot\left[P(R)\right]^X\cdot\left[P(R^C)\right]^{n-X}\\
P(\mathcal{W}_3) &= C(10,3)\cdot(0.3)^3\cdot(0.7)^{10-3}\\
P(\mathcal{W}_3) &= 120\cdot(0.027)\cdot(0.7)^7\\
P(\mathcal{W}_3) &\approx 0.2668279
\end{aligned}
$$

We are now ready to answer the question, "What is the probability, that it will rain at most three times in ten instances of these weather conditions?" In other words, "What is the value of $P(\mathcal{W}_1\cup\mathcal{W}_2\cup\mathcal{W}_3)$?" Applying the *Addition Rule of Probability for Mutually Exclusive Events* we find $P(\mathcal{W}_1\cup\mathcal{W}_2\cup\mathcal{W}_3)$ below.

$$
\begin{aligned}
P(\mathcal{W}_1\cup\mathcal{W}_2\cup\mathcal{W}_3) &= P(\mathcal{W}_1)+P(\mathcal{W}_2)+P(\mathcal{W}_3)\\
P(\mathcal{W}_1\cup\mathcal{W}_2\cup\mathcal{W}_3) &\approx 0.1210608+0.2334744+0.2668279\\
P(\mathcal{W}_1\cup\mathcal{W}_2\cup\mathcal{W}_3) &\approx 0.6213631
\end{aligned}
$$

The mean of the binomial distribution is equal to the product of *n* and *p*.

The *mean of the binomial distribution* of sample size *n* with *p* probability of success is given by

$$\mu = E(X) = np\,.$$

The standard deviation of the binomial distribution equals the square root of the product of the mean and the probability of failure.

The *standard deviation of the binomial distribution* of sample size *n* with *p* probability of success is given by

$$\sigma = \sqrt{np(1-p)} = \sqrt{\mu(1-p)}\,.$$

Instruction: The Poisson Probability Distribution

Suppose that we want to find the probability distribution of the number of industrial accidents at a particular station during a time period of one month. Initially, this random variable may not seem related to a binomial random variable, but there is an interesting relationship.

Think of the time period as being split into *n* subintervals, each of which is so small that at most one accident could occur in it. Denoting the probability of an accident in any subinterval by *p*, we have a binomial probability because the probability that one accident occurs equals *p* while the probability that no accident occurs equals 1 – *p* (and the probability that more than one accident occurs equals zero). Using this relationship to the binomial distribution, it can be shown that the probability of *X*-number of industrial accidents in one month at a particular station is given by $P(X) = \frac{e^{-\lambda}\lambda^X}{X!}$ where lambda equals the expected number of industrial accidents, i.e., the mean. Random variables possessing this distribution are Poisson random variables.

In general, the number of random events that occur in an interval of time or space or any other dimension often follows the Poisson distribution. The Poisson distribution applies particularly to "rare" events, that is, events which occur infrequently in time, space, volume, etc. The dimension in which the events occur is called the *area of opportunity*.

The Poisson probability distribution where λ events are expected is given by

$$P(X) = \frac{e^{-\lambda} \cdot \lambda^X}{X!}$$

where $P(X)$ equals the probability of *X* events in an area of opportunity.

Consider a case where the mean number of cars pulling into the drive-thru lane of a restaurant during a two-minute interval is two. The probability that the number of vehicles exceeds three can be calculated using the Poisson distribution. First, we calculate the probability of exactly zero cars entering the lane as shown below.

$$P(X=0) = \frac{e^{-2} \cdot 2^0}{0!}$$

$$P(X=0) \approx 0.1353$$

Second, we calculate the probability of exactly one car entering the lane as well as exactly two cars, and exactly three cars.

$$P(X=1)=\frac{e^{-2}\cdot 2^{1}}{1!}, \quad P(X=2)=\frac{e^{-2}\cdot 2^{2}}{2!}, \quad P(X=3)=\frac{e^{-2}\cdot 2^{3}}{3!}$$
$$P(X=1)\approx 0.2707 \quad P(X=2)\approx 0.2707 \quad P(X=3)\approx 0.1804$$

To find the probability that the number of cars exceeds three, we use the Complement Principle as below.

$$P(X>3)=1-\left[P\left((X=0)\cup(X=1)\cup(X=2)\cup(X=3)\right)\right]$$
$$P(X>3)=1-\left[P(X=0)+P(X=1)+P(X=2)+P(X=3)\right]$$
$$P(X>3)\approx 1-\left[0.1353+0.2707+0.2707+0.1804\right]$$
$$P(X>3)\approx 1-[0.8571]$$
$$P(X>3)\approx 0.1429$$

Problems

#1 An experiment requires the toss of a four-sided die and a six-sided die and observing the *sum* of the results. Construct a probability distribution for the experiment.

#2 Consider a game where the player spins the number wheel in Figure A and loses a dollar amount equal to odd values indicated by the arrow but wins a dollar amount equal to ten times even values indicated by the arrow. What is the expected value of the game?

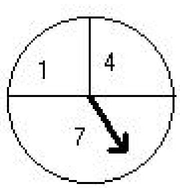


Figure A

#3 An insurance agency sells an insurance policy that pays $50,000.00 in benefits in a given year if a worker is injured on the job and stays in the hospital one week or more. The policy sells for a monthly premium of $42.00. Qualified workers must work for a company whose safety regulations give the workers only a 0.002 probability of injury. What is the expected value of the policy for the insurance agency?

#4 Consider an experiment that requires spinning the dial in Figure A. What is the probability that the arrow will indicate the number seven exactly five times in nine trials?

#5 The mean number of vehicles entering a tunnel per two-minute interval is one. Find the probability that the number of vehicles entering the tunnel during a two minute period exceeds three.

Bonus Assume that the tunnel from problem five is watched during ten two-minute intervals, thus giving ten independent observations, $X_1, X_2, \ldots, X_{10}$, on the Poisson random variable. Find the probability that $X > 3$ exactly two times.