Calculus
Practice Test 3

Use $f(x) = x^3 - 3x^2 - 24x + 32$ for problems #1 – 3.

#1) Find all relative extrema for $f(x)$.

#2) Find all the inflections points for $f(x)$.

#3) Find the absolute minimum for $f(x)$ along the interval (-5,5].

#4) If $g(x) = e^{-x^2}$, find the intervals where $g(x)$ is concave upward.

#5) $\int \frac{t^3 + 2t^2}{\sqrt{t}}$

#6) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin(t)dt$

#7 $\lim_{x \to 2} \frac{(x-2)}{(x^2-4)}$

#8) $\lim_{x \to 1} x^{\frac{1}{1-x}}$

#9) Eleven thousand spectators attend the local hockey game when the ticket price is set at \$12 and twelve thousand spectators attend when the ticket price is set at \$11. What price maximizes revenue?

#10) Approximate the area under $h(x) = -2^{-x} + 4$ along [-2,2] using 4 rectangles.

Practice Test 3

SOLUTIONS

Use $f(x) = x^3 - 3x^2 - 24x + 32$ for problems #1 – 3.

#1) Find all relative extrema for $f(x)$.

$f(x) = x^3 - 3x^2 - 24x + 32$
$f'(x) = 3x^2 - 6x - 24$
$3x^2 - 6x - 24 = 0$
$3(x^2 - 2x - 8) = 0$
$3(x + 2)(x - 4) = 0$
$x + 2 = 0 \qquad x - 4 = 0$
$x = -2 \qquad x = 4$

Extrema can occur where the first derivative is undefined or where the first derivative is equal to zero. Wherever the first derivative is zero is where the tangent line is horizontal, and horizontal lines have a slope of zero.

$f'(-3) = 3(-3)^2 - 6(-3) - 24$
$f'(-3) = 21$
$f'(-1) = 3(-1)^2 - 6(-1) - 24$
$f'(-1) = -15$
$f'(5) = 3(5)^2 - 6(5) - 24$
$f'(5) = 21$

A function increases wherever its derivative is positive and decreases wherever its derivative is negative. $f(x)$ is increasing at x = -3 but decreasing at x = -1, so $f(x)$ has a relative maximum at x = -2. $f(x)$ is increasing again at x = 5, so the function has a relative minimum at x = 4.

$f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 32$
$f(-2) = 60$
$f(4) = (4)^3 - 3(4)^2 - 24(4) + 32$
$f(4) = -48$

$f(x)$ has a relative maximum at the point $f(-2) = 60$.
$f(x)$ has a relative minimum at the point $f(4) = -48$.

#2) Find all the inflections points for $f(x)$.

$f(x) = x^3 - 3x^2 - 24x + 32$
$f'(x) = 3x^2 - 6x - 24$
$f''(x) = 6x - 6$
$6x - 6 = 0$
$6x = 6$
$x = 1$

Inflection can occur wherever the second derivative is equal to zero.

Inflection occurs where a function changes in concavity. A function is concave down wherever the second derivative is negative and concave up wherever the second derivative is positive. $f(x)$ is concave down at x = 0 and concave up at x = 2; therefore, inflection occurs at x = 1.

$f''(0) = 6(0) - 6$
$f''(0) = -6$
$f''(2) = 6(2) - 6$
$f''(2) = 6$

$f(x)$ has an inflection point at $f(1) = 6$.

$f(1) = (1)^3 - 3(1)^2 - 24(1) + 32$
$f(1) = 6$

#3) Find the absolute minimum for $f(x)$ along the interval (-5,5].

From problem one, we know that $f(x)$ has a relative minimum at the point $f(4) = -48$. This point could be a absolute minimum along the interval (-5,5]. Also from problem one we know that $f(x)$ is increasing along the interval (4,5) and decreasing along (-2,4); therefore, any points lower than $f(4) = -48$ along the given interval must occur along (-5,-2). Since the function is increasing along this interval, the lowest point will occur at the beginning of the interval where x = -5. Since $f(-5) = -48$, then all the other points to its right are larger and $f(4) = -48$ is the absolute minimum on the given interval.

$$f(-5) = (-5)^3 - 3(-5)^2 - 24(-5) + 32$$
$$f(-5) = -48$$

#4) If $g(x) = e^{-x^2}$, find the intervals where $g(x)$ is concave upward.

A function is concave upward where its second derivative is positive (refer to problem two). To find intervals of upward concavity, first find the second derivative. Second, set the second derivative equal to zero. Test x-values left and right of the x-values where the second derivative is zero to find where it is positive or negative.

$$g(x) = e^{-x^2}$$
$$g'(x) = e^{-x^2} \cdot \frac{d}{dx} - x^2$$
$$g'(x) = e^{-x^2} \cdot -2x$$
$$g'(x) = -2xe^{-x^2}$$

$$g''(x) = -2x \cdot \frac{d}{dx} e^{-x^2} + e^{-x^2} \cdot \frac{d}{dx} - 2x$$
$$g''(x) = -2x \cdot -2xe^{-x^2} + e^{-x^2} \cdot -2$$
$$g''(x) = 4x^2 e^{-x^2} - 2e^{-x^2}$$
$$g''(x) = 2e^{-x^2}(2x^2 - 1)$$

Factor out the greatest common factor from the binomial.

Set the factors equal to zero. Realize that e raised to any power will never be zero, so it is only necessary to set the second factor equal to zero.

$$2x^2 - 1 = 0$$
$$2x^2 = 1$$
$$x^2 = ½$$
$$\sqrt{x^2} = \sqrt{\frac{1}{2}}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

Since –4, 0, and 4 are values that fall on the left and right of $\pm\frac{1}{\sqrt{2}}$ then evaluating $g''(-4)$, $g''(0)$, and $g''(4)$ will determine the intervals where the second derivative is positive and will thereby determine where the function is concave upward.

$$g''(-4) \approx .000006977$$
$$g''(0) \approx -2$$
$$g''(4) \approx .000006977$$

The function is concave upward along the interval $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$.

#5)

$$\int \frac{t^3 + 2t^2}{\sqrt{t}}$$

$$\int \frac{t^3 + 2t^2}{t^{\frac{1}{2}}}$$

$$\int t^{-\frac{1}{2}}\left(t^3 + 2t^2\right)$$

$$\int t^{\frac{5}{2}} + 2t^{\frac{3}{2}}$$

$$\frac{2}{7}t^{\frac{7}{2}} + 2\cdot\frac{2}{5}t^{\frac{5}{2}}$$

$$\boxed{\frac{2}{7}t^{\frac{7}{2}} + \frac{4}{5}t^{\frac{5}{2}}}$$

#6)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin(t)dt$$

$$-\cos(t) + C - [-\cos(t) + C]\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$-\cos\left(\frac{\pi}{3}\right) + C - \left[-\cos\left(\frac{\pi}{4}\right) + C\right]$$

$$-\left(-\frac{1}{2}\right) + C - \left[-\frac{1}{\sqrt{2}} + C\right]$$

$$\frac{1}{2} + C + \frac{1}{\sqrt{2}} - C$$

$$\frac{1}{2} + \frac{1}{\sqrt{2}}$$

$$\frac{\sqrt{2}}{2\sqrt{2}} + \frac{2}{2\sqrt{2}}$$

$$\frac{\sqrt{2} + 2}{2\sqrt{2}}$$

$$\frac{\sqrt{2} + 2}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\frac{2 + 2\sqrt{2}}{4}$$

$$\frac{2\left(1 + \sqrt{2}\right)}{4}$$

$$\boxed{\frac{\left(1 + \sqrt{2}\right)}{2}}$$

#7 $\lim_{x\to 2} \frac{(x-2)}{(x^2-4)}$

Since the limit is indeterminate, $(\frac{0}{0})$, use L'Hospital's rule.

$$\lim_{x\to 2} \frac{\frac{d}{dx}(x-2)}{\frac{d}{dx}(x^2-4)}$$

$$\lim_{x\to 2} \frac{1}{2x}$$

$$\frac{1}{2\cdot 2} = \frac{1}{4}$$

$\lim_{x\to 2} \frac{(x-2)}{(\ ^2 -\ )} = ¼$

#8) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

$$\ln \lim = \lim_{x\to 1} \ln x^{\frac{1}{1-x}}$$

Take the natural log of the limit. Use logarithmic properties to eliminate the rational exponent.

$$\ln \lim = \lim_{x\to 1} \frac{1}{1-x} \ln x$$

$$\ln \lim = \lim_{x\to 1} \frac{\ln x}{1-x}$$

Applying the limit and substituting in the one, shows that the limit is indeterminate, so use L'Hospital's rule.

$$\ln \lim = \lim_{x\to 1} \frac{\frac{d}{dx}\ln x}{\frac{d}{dx}1-x}$$

$$\ln \lim = \lim_{x\to 1} \frac{\frac{1}{x}}{-1}$$

$$\ln \lim = \frac{\frac{1}{1}}{-1}$$

Apply the limit by substituting in the one then simplify.

$$\ln \lim = -1$$

$$e^{\ln \lim} = e^{-1}$$

$\lim_{x\to 1} = \frac{1}{e}$

#9) Eleven thousand spectators attend the local hockey game when the ticket price is set at $12 and twelve thousand spectators attend when the ticket price is set at $11. What price maximizes revenue?

Since demand is a function of the price of a product, the information given can be written as ordered pairs with prices as *x*-values and attendance as the demand, the *y*-values.

$D(12) = 11{,}000$
$D(11) = 12{,}000$

$$m = \frac{12{,}000 - 11{,}000}{11 - 12} = \frac{1{,}000}{-1} = -1{,}000$$

Assuming that demand is linear, we can begin to find the demand function by deriving its rate of change per unit change in price (its slope).

$$y - y_1 = m(x - x_1)$$
$$y - 11{,}000 = -1{,}000(x - 12)$$
$$y - 11{,}000 = -1{,}000x + 12{,}000$$
$$y = -1{,}000x + 23{,}000$$
$$D(x) = -1{,}000x + 23{,}000$$

With the slope of the linear demand function, we can use one of the ordered pairs to find the demand function.

$$R(x) = demand \cdot price$$
$$R(x) = (-1{,}000x + 23{,}000) \cdot x$$
$$R(x) = -1{,}000x^2 + 23{,}000x$$

Revenue is a function of the price of the product times the number of units sold, i.e., the demand.

$R'(x) = -2{,}000x + 23{,}000$
$-2{,}000x + 23{,}000 = 0$
$-2{,}000x = -23{,}000$
$x = 11.5$

Extrema occur where the first derivative equals zero, so we can find the price that creates an extrema in the revenue function by finding where the first derivative is equal to zero.

$R'(11) = -2{,}000(11) + 23{,}000$
$R'(11) = 1{,}000$
$R'(12) = -2{,}000(12) + 23{,}000$
$R'(12) = -1{,}000$

A maximum occurs where the function changes from increasing to decreasing. A function increases wherever its derivative is positive and decreases wherever its derivative is negative. Since the function increases along $(-\infty, 11.5)$ and decreases along $(11.5, \infty)$, we know 11.5 is the x-value that maximizes the function.

$R(x) = -1{,}000x^2 + 23{,}000x$
$R(11.5) = -1{,}000(11.5)^2 + 23{,}000(11.5)$
$R(11.5) = 132{,}250$

The ticket price of $11.50 will maximize the revenue at $132,250.00.

#10) Approximate the area under $h(x) = -2^{-x} + 4$ along [-2,2] using 4 rectangles.

$2 - -2 = 4$

Find the length of the interval (the difference between the x-values.

$4/4 = 1$

Divide the length by four to find the width of the four rectangles. This creates four sub-intervals [-2,-1], [-1,0],[0,1],[1,2] with mid-points at -1.5, -.5, .5, & 1.5.

$h(-1.5) = -2^{-1.5} + 4$
$h(-1.5) \cong 1.17$
$h(-.5) = -2^{-.5} + 4$
$h(-.5) \cong 2.59$
$h(.5) = -2^{.5} + 4$
$h(.5) \cong 3.29$
$h(1.5) = -2^{1.5} + 4$
$h(1.5) \cong 3.65$

Graph the function. Find the value of the function at the mid-point of each of these intervals. Draw a horizontal line along each interval that cuts through the function at these mid-points. Draw the four rectangles using the mid-point values as the heights of the rectangles and the interval length as the widths of the rectangles. Find the area of each rectangle using these heights and widths. Add each of the resulting areas together to approximate the area under the curve.

$A = h \cdot w$	$A = h \cdot w$	$A = h \cdot w$	$A = h \cdot w$
$A \cong 1.17 \cdot 1$	$A \cong 2.59 \cdot 1$	$A \cong 3.29 \cdot 1$	$A \cong 3.65 \cdot 1$
$A \cong 1.17$	$A \cong 2.59$	$A \cong 3.29$	$A \cong 3.65$

Total Area $\cong 1.17 + 2.59 + 3.29 + 3.65$

Total Area $\cong$ 10.7 square units