

Calculus Practice Test 2

#1) Find y' if $x^3 + y^3 = 6xy$.

#2) If A is the area of a circle with radius r , find dA/dt in terms of dr/dt .

#3) Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$, and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

#4) For the problem above identify dy and Δy .

#5) If $s(t) = 2t^3 - 5t^2 + 3t + 4$, find $s''(t)$.

#6) Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4-x)$.

#7) Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$ with domain $[-\frac{1}{2}, 4]$.

#8) Differentiate $f(x) = \sqrt{\ln x}$.

#9) Differentiate a] $y = \sin(x^2)$ and b] $y = \sin^2 x$

#10) Differentiate $y = \frac{x^{3/4} \sqrt{x^2 + 1}}{(3x + 2)^5}$.

SOLUTIONS

#1) Find y' if $x^3 + y^3 = 6xy$.

$$\frac{d}{dx} x^3 + \frac{d}{dx} y^3 = \frac{d}{dx} 6xy$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x \cdot \frac{dy}{dx} + y \cdot 6$$

$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = -3x^2 + 6y$$

$$\frac{dy}{dx} (3y^2 - 6x) = -3x^2 + 6y$$

$$\frac{dy}{dx} = \frac{-3x^2 + 6y}{(3y^2 - 6x)}$$

$$\frac{dy}{dx} = \frac{3(-x^2 + 2y)}{3(y^2 - 2x)}$$

$$\boxed{y' = \frac{-x^2 + 2y}{y^2 - 2x}}$$

#2) If A is the area of a circle with radius r , find dA/dt in terms of dr/dt .

$$A = \pi r^2$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

$$\boxed{\frac{dA}{dt} = 2\pi r \frac{dr}{dt}}$$

#3) Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$, and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

$$f(x) = (x+3)^{1/2}$$

$$f'(x) = \frac{1}{2}(x+3)^{-1/2} \cdot 1$$

$$f'(x) = \frac{1}{2\sqrt{x+3}}$$

First, find the derivative of the given function.

$$f(1) = \sqrt{1+3} = 2$$

$$f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$

Second, evaluate $f(1)$ to find the corresponding y -value for the given x -value ($a=1$). Also, evaluate $f'(1)$ to find the slope of the tangent line at the given x -value.

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = f(1) + f'(1)(x-1)$$

$$L(x) = 2 + \frac{1}{4}(x-1)$$

$$L(x) = 2 + \frac{1}{4}x - \frac{1}{4}$$

$$L(x) = \frac{1}{4}x + \frac{7}{4}$$

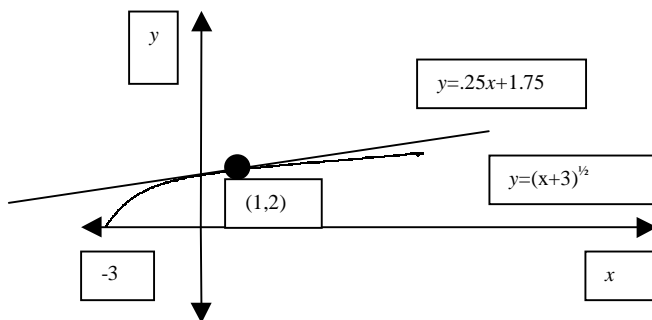
Third, point-slope formula to find the linearization that can be used to approximate values of the function.

$$\sqrt{3.98} \approx \frac{1}{4}(.98) + \frac{7}{4} \approx 1.995$$

$$\sqrt{4.05} \approx \frac{1}{4}(1.05) + \frac{7}{4} \approx 2.0125$$

Finally, use the linearization to approximate the given numbers. Remember, that the function adds 3 to the x -values.

The linear approximations above could have been supplied by a calculator. The linearization, however, gives approximations over an entire interval as can be seen in the illustration below.



#4) For the problem above identify dy and Δy .

$$\Delta y = \frac{1}{4}x + \frac{7}{4}$$

$$dy = \frac{1}{2\sqrt{x+3}}$$

Δy represents the rate of change as given by the line tangent to the function at the point (1,2).

dy represents the rate of change on the function.

#5) If $s(t) = 2t^3 - 5t^2 + 3t + 4$, find $s''(t)$.

$$s'(t) = 6t^2 - 10t + 3$$

$$s''(t) = 12t - 10$$

#6) Find the critical numbers of $f(x) = x^{\frac{3}{5}}(4-x)$.

$$f'(x) = \frac{3}{5}x^{-\frac{2}{5}}(4-x) + x^{\frac{3}{5}}(-1)$$

$$f'(x) = \frac{3(4-x)}{5x^{\frac{2}{5}}} - x^{\frac{3}{5}}$$

$$f'(x) = \frac{12-3x}{5x^{\frac{2}{5}}} - \frac{5x}{5x^{\frac{2}{5}}}$$

$$f'(x) = \frac{12-8x}{5x^{\frac{2}{5}}}$$

Find the derivative using the product rule.

$$12-8x = 0$$

$$-8x = -12$$

$$x = 1.5$$

$$5x^{\frac{2}{5}} = 0$$

$$x^{\frac{2}{5}} = 0$$

$$\left(x^{\frac{2}{5}}\right)^{\frac{5}{2}} = (0)^{\frac{5}{2}}$$

$$x = 0$$

Critical numbers are the values that make the derivative equal to zero or undefined. This derivative will equal zero where its numerator equals zero. Consequently, 1.5 is a critical number. This derivative will be undefined where its denominator equals zero. Consequently, zero is also a critical number.

The critical numbers are 1.5 and 0.

#7) Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$ with domain $[-\frac{1}{2}, 4]$.

$$f'(x) = 3x^2 - 6x$$

$$3x^2 - 6x = 0$$

$$3x(x - 2) = 0$$

$$x = 0 \text{ \& } 2$$

Extrema occur where the derivative equals zero, so we should begin by finding the critical numbers in the interval.

$$f'(-.1) = 3(-.1)^2 - 6(-.1)$$

$$f'(-.1) = .63$$

$$f'(1) = 3(1)^2 - 6(1)$$

$$f'(1) = -.57$$

$$f'(3) = 3(3)^2 - 6(3)$$

$$f'(3) = 9$$

Testing values left and right of the critical numbers in the derivative reveals where the function is increasing and decreasing. Since the derivative is positive left of zero and right of two but negative between zero and two, the function increases $(-\frac{1}{2}, 0) \cup (2, 4)$ and decreases $(0, 2)$.

$$f(0) = (0)^3 - 3(0)^2 + 1$$

$$f(0) = 1$$

$$f(2) = (2)^3 - 3(2)^2 + 1$$

$$f(2) = -3$$

Since the function changes behavior at the critical numbers, as seen above, the function has extrema at those x -values. Evaluating the function using the critical numbers yields the extrema: $f(0) = 1$, a relative maximum, and $f(2) = -3$, a relative minimum.

$$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 + 1$$

$$f\left(-\frac{1}{2}\right) = \frac{1}{8}$$

$$f(4) = (4)^3 - 3(4)^2 + 1$$

$$f(4) = 17$$

The end points of the interval also represent extrema. Evaluating the function using the x -values at the ends of the intervals yields the extrema: $f(-\frac{1}{2}) = \frac{1}{8}$, a minimum, and $f(4) = 17$, a maximum.

Comparing all the maximums and minimums along the interval reveals the absolute extrema. Since $f(2) < f(-\frac{1}{2})$, the absolute minimum is $f(2) = -3$. Since $f(4) > f(0)$, the absolute maximum is $f(4) = 17$.

#8) Differentiate $f(x) = \sqrt{\ln x}$.

$$f(x) = (\ln x)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot \frac{d}{dx} \ln x$$

$$f'(x) = \frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x}$$

Use the chain rule.

$$f'(x) = \frac{1}{2x\sqrt{\ln x}}$$

#9) Differentiate a] $y = \sin(x^2)$ and b] $y = \sin^2 x$

Solution to Part A

$$y' = \cos(x^2) \cdot \frac{d}{dx} x^2$$

$$y' = \cos(x^2) \cdot 2x$$

$$y' = 2x \cos(x^2)$$

Remember the rule:

$$y = \sin[f(x)]$$

$$y' = \cos[f(x)] \cdot f'(x)$$

Solution to Part B

$$y = (\sin x)^2$$

$$y' = 2(\sin x)^1 \cdot \frac{d}{dx} \sin x$$

$$y' = 2(\sin x) \cdot (\cos x)$$

Use the chain rule.

$$y' = 2 \sin x \cdot \cos x$$

#10) Differentiate $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$.

Take the natural log of both sides.

$$\ln y = \ln \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$$

Use logarithmic properties to expand the expression.

$$\ln y = \ln x^{\frac{3}{4}}\sqrt{x^2+1} - \ln(3x+2)^5$$

$$\ln y = \ln x^{\frac{3}{4}} + \ln(x^2+1)^{\frac{1}{2}} - \ln(3x+2)^5$$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

Differentiate implicitly.

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{(x^2+1)} \cdot \frac{d}{dx}(x^2+1) - 5 \cdot \frac{1}{(3x+2)} \cdot \frac{d}{dx}(3x+2)$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{1}{(x^2+1)} \cdot 2x - 5 \cdot \frac{1}{(3x+2)} \cdot 3$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{4x} + \frac{1}{2(x^2+1)} \cdot 2x - \frac{5}{(3x+2)} \cdot 3$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{3}{4x} + \frac{2x}{2(x^2+1)} - \frac{15}{(3x+2)}$$

Solve for dy/dx .

$$y \cdot \left(\frac{1}{y} \frac{dy}{dx} \right) = y \cdot \left(\frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{(3x+2)} \right)$$

Substitute original function in for y .

$$\frac{dy}{dx} = y \cdot \left(\frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{(3x+2)} \right)$$

$$\frac{dy}{dx} = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5} \cdot \left(\frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{(3x+2)} \right)$$