

## DETERMINING THE BEHAVIOR OF FUNCTIONS USING DERIVATIVES

Practice Problem for  
Calculus, Math 2413,  
and  
Math for Business & Social Sciences, Math 1325

The following is a nine-step process to sketching a function using information derived from the function's derivatives. After reading, the student should try using the nine-step process on his or her own.

Given:  $f(x) = x^3 - 3x^2 - 24x + 32$

1. Find the critical numbers of  $f(x)$ .

$$f(x) = x^3 - 3x^2 - 24x + 32$$

$$f'(x) = 3x^2 - 6x - 24$$

$$3x^2 - 6x - 24 = 0$$

$$3(x^2 - 2x - 8) = 0$$

$$(x - 4)(x + 2) = 0$$

$$x - 4 = 0, x + 2 = 0$$

$$x = 4, -2$$

The critical numbers are the x-values where the derivative equals zero or is undefined.

2. Find the intervals where  $f(x)$  increases and decreases.

$$f'(x) = 3x^2 - 6x - 24$$

$$f'(-3) = 3(-3)^2 - 6(-3) - 24$$

$$f'(-3) = 21$$

The function increases from negative infinity up to  $-2$ .

$$f'(x) = 3x^2 - 6x - 24$$

$$f'(0) = 3(0)^2 - 6(0) - 24$$

$$f'(0) = -24$$

The function decreases from  $-2$  to  $4$ .

$$f'(x) = 3x^2 - 6x - 24$$

$$f'(5) = 3(5)^2 - 6(5) - 24$$

$$f'(5) = 21$$

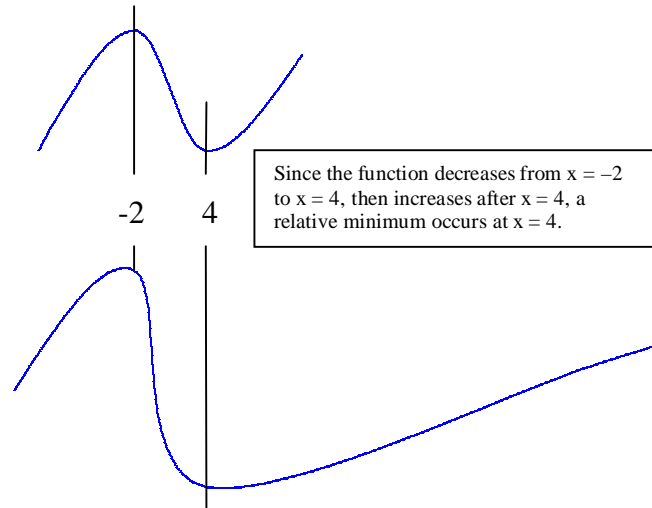
The function increases from  $4$  to positive infinity.

Wherever the derivative is negative, the function is decreasing, and wherever the derivative is positive the function is increasing. Consequently, to find intervals of increasing and decreasing behavior in a function, one must find where the derivative is positive and negative. The critical numbers  $-2$  &  $4$  represent where the derivative of  $f(x)$  equals zero; therefore, the derivative is positive and or negative to the left and right of these numbers. Test values left and right of each critical number to find where the derivative is positive and negative.

The function increases along the intervals  $(-\infty, -2) \cup (4, \infty)$  and decreases along the interval  $(-2, 4)$ .

3. Using the information gained from the previous two steps, sketch a rough approximation of the graph of  $f(x)$  without drawing any axes.

Since the function increases until  $x = -2$  then decreases, a relative maximum must occur at  $x = -2$ .



Since the function decreases from  $x = -2$  to  $x = 4$ , then increases after  $x = 4$ , a relative minimum occurs at  $x = 4$ .

Both of these drawings match information gathered so far. Your sketch may vary from these.

4. Find all extrema.

$$f(x) = x^3 - 3x^2 - 24x + 32$$

$$f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 32$$

$$f(-2) = 60$$

relative maximum:  $f(-2) = 60$

$$f(4) = (4)^3 - 3(4)^2 - 24(4) + 32$$

$$f(4) = -48$$

relative minimum:  $f(4) = -48$

5. Find the critical numbers of the derivative.

$$f'(x) = 3x^2 - 6x - 24$$

$$f''(x) = 6x - 6$$

The critical numbers of the derivative will be where its derivative (the second derivative of the function) is undefined or equal to zero.

$$6x - 6 = 0$$

$$6x = 6$$

$$x = 1$$

## 6. Find the function's intervals of concavity.

A function is concave down wherever its second derivative is negative and concave up wherever its second derivative is positive. From the critical number of the second derivative above,  $x = 1$ , we know the second derivative will be positive or negative to the right and left of  $x = 1$ . Consequently, testing the second derivative with values left and right of  $x = 1$  will show where the second derivative is pos or neg, thereby revealing where the intervals of concavity on the function.

$$f''(x) = 6x - 6$$

$$f''(0) = 6(0) - 6$$

$$f''(0) = -6$$

The function is concave down from negative infinity to 1.

$$f''(2) = 6(2) - 6$$

$$f''(2) = 6$$

The function is concave up from 1 to positive infinity.

The function is concave down along the interval  $(-\infty, 1)$ .

The function is concave up along the interval  $(1, \infty)$ .

## 7. Find the function's point of inflection.

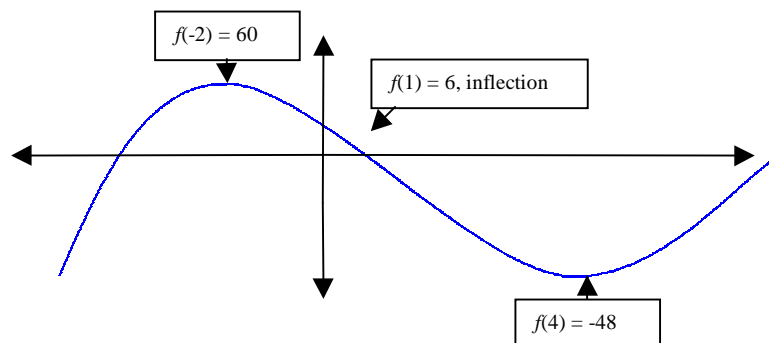
$$f(x) = x^3 - 3x^2 - 24x + 32$$

$$f(1) = 1^3 - 3(1)^2 - 24(1) + 32$$

$$f(1) = 6$$

Since a change in type of concavity occurs at  $x = 1$ , inflection occurs at  $x = 1$ .

The inflection point is  $f(1) = 6$ .

8. Graph the function with a domain of  $(-5, 6]$ . Set your window to include the range  $(-100, 100)$  with a scale of 10 on the y-axes.

Without a graphing utility, more analysis would be needed to determine the value of the function at the end-points. This is done in the next step, finding absolute extrema.

9. Find the absolute extrema of  $f(x)$  in the domain of  $(-5,6]$ .

$$f(x) = x^3 - 3x^2 - 24x + 32$$

$$f(-5) = (-5)^3 - 3(-5)^2 - 24(-5) + 32$$

$$f(-5) = -48$$

Remember that a parenthesis indicates that a given bound is not included in the interval. Since  $f(-5) = -48$  is not actually included in the domain  $(-5,6]$  and since all values to its immediate right will be greater than  $-48$  (due to the function's increasing behavior),  $f(4) = -48$  is the absolute minimum of the function with domain  $(-5,6]$ .

$$f(6) = (6)^3 - 3(6)^2 - 24(6) + 32$$

$$f(6) = -4$$

The graph shows that  $f(-2) = 60$  is the absolute maximum of the function with domain  $(-5,6]$ . Finding  $f(6) = -4$  verifies that the end-point along the next interval of increasing behavior does not reach a value greater than 60.

Absolute maximum:  $f(-2) = 60$

Absolute minimum:  $f(4) = -48$

Practice Exercise: Construct a three-degree polynomial by selecting three linear factors from the list below. After constructing the polynomial, follow the provided example and perform each of the nine steps with the constructed polynomial function. A graphing utility can verify answers.

Linear factors:  $(3x + 10)$ ,  $(3x+14)$ ,  $(4x + 9)$ ,  $(2x + 13)$ ,  $(2x + 9)$ ,  $(2x + 15)$ ,  
 $(6x + 25)$ ,  $(5x - 19)$ ,  $(3x - 10)$ ,  $(4x - 19)$ ,  $(4x - 23)$ ,  $(7x - 25)$ ,  
 $(7x - 9)$ ,  $(5x - 12)$ ,  $(5x - 22)$ ,  $(2x - 17)$ ,  $(6x + 19)$ ,  $(x + 4)$

To construct a three-degree polynomial simply choose three binomials above and multiply them together.