Math 1325
Practice Test #2

#1) Find the intervals where $f(x)$ is increasing and decreasing.

$f(x) = 4x^2 + 16x$

#2) Find the relative maxima and relative minima of $g(x)$.

$g(x) = x^3 - 3x^2 - 24x + 32$

#3) Find the absolute maximum and absolute minimum of $h(x)$ in the interval [-2,2].

$h(x) = 4 - x^2$

#4) Find the points of inflection of $s(x)$.

$s(x) = x^3$

#5 Sketch the graph labeling extrema and inflection points:

$h(x) = e^{-x^2}$ on [-2,2].

#6 Describe the graph of $f(x) = x^4 - 2x^2$.

#7 Describe the graph of $h(x) = x^2 - \ln x$

SOLUTIONS

#1) Find the intervals where $f(x)$ is increasing and decreasing.

$f(x) = 4x^2 + 16x$
$f'(x) = 8x + 16$
$8x + 16 = 0$
$8x = -16$
$x = -2$

Begin by finding the critical numbers (the x-values where a function's derivative equals zero.

$f'(-3) = 8(-3) + 16$
$f'(-3) = -8$
$f'(0) = 8(0) + 16$
$f'(0) = 16$

If the derivative changes sign at the critical number, then the critical number represents a turning point, an extrema, in the function. Since $f'(x)$ is negative left of $x = -2$ and positive to the right of $x = -2$, the function changes behavior at $x = -2$.

INCREASING $(-2, \infty)$
DECREASING $(-\infty, -2)$

A function increases wherever its derivative is positive and decreases wherever its derivative is negative, so $f(x)$ increases along the interval $(-2, \infty)$ and decreases along the interval $(-\infty, -2)$.

#2) Find the relative maxima and relative minima of $g(x)$.

$g(x) = x^3 - 3x^2 - 24x + 32$
$g'(x) = 3x^2 - 6x - 24$
$3x^2 - 6x - 24 = 0$
$x^2 - 2x - 8 = 0$
$(x + 2)(x - 4) = 0$
$x + 2 = 0 \quad x - 4 = 0$
$x = -2, \ 4$

Extrema can occur where the first derivative is undefined and/or equal to zero, so begin by finding the critical numbers.

$g'(-3) = 3(-3)^2 - 6(-3) - 24$
$g'(-3) = 21$
$g'(0) = 3(0)^2 - 6(0) - 24$
$g'(0) = -24$
$g'(5) = 3(5)^2 - 6(5) - 24$
$g'(5) = 21$

Testing to the left and right of the critical numbers to see if the derivative changes sign determines if the critical numbers represent the x-values of extrema. If the derivative changes sign, the critical number will be an extrema.

$g(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 32$

$g(-2) = 60$ relative maxima

$g(-2) = 60$ is a maximum because the function increases up to -2 then decreases.

$g(4) = (4)^3 - 3(4)^2 - 24(4) + 32$

$g(4) = -48$ relative

$g(4) = -48$ is a minimum because the function decreases up to 4 then increases.

#3) Find the absolute maximum and absolute minimum of $h(x)$ in the interval [–2,2].

$h(x) = 4 - x^2$
$h'(x) = -2x$
$-2x = 0$
$x = 0$

Extrema can occur where the derivative is undefined and/or equal to zero. Begin by finding derivative. Set derivative equal to zero and solve. Zero is a critical number and possibly an extrema.

$h'(-1) = -2(-1) = 2$
$h'(1) = -2(1) = -2$

This shows that the function changes from increasing to decreasing at $x = 0$.

$h(0) = 4 - 0^2$
$h(0) = 4$

The function increases (-2,0) and decreases (0,4), so the highest point must occur at $x = 0$. Consequently, $h(0) = 4$ is an absolute maximum.

$h(-2) = 4 - (-2)^2$
$h(-2) = 0$
$h(2) = 4 - (2)^2$
$h(-2) = 0$

The function increases (-2,0) and decreases (0,4), so the lowest pair of points must occur at $x = -2$ and $x = 2$. Since the function has the same value at $x = -2$ and $x = 2$, both points represent absolute minimums.

ABSOLUTE MAXIMUM: $h(0) = 4$
ABSOLUTE MINIMUM: $h(-2) = 0$ and $h(2) = 0$

#4) Find the points of inflection of $s(x)$.

$s(x) = x^3$
$s'(x) = 3x^2$
$s''(x) = 6x$

Inflection is where a function changes concavity. Inflection points occur where the second derivative is equal to zero, so begin by finding the second derivative.

$6x = 0$
$x = 0$

Zero could be where inflection takes place if the function changes concavity.

$s''(-1) = 6(-1) = -6$

$s''(1) = 6(1) = 6$

A function is concave down where its second derivative is negative and concave up where its second derivative is positive. So, $s(x)$ is concave up before x=0 and concave up after x=0.

$s(x) = x^3$
$s(0) = 0^3$
$s(x) = 0$

Since the function changes in concavity at x=0, zero is the x-value where inflection occurs in the function. Evaluating the function at x=0 will give the inflection point.

INFLECTION POINT $s(0) = 0$

#5 Sketch the graph labeling extrema and inflection points:

$h(x) = e^{-x^2}$ on [–2,2].

$h'(x) = -2xe^{-x^2}$

$-2xe^{-x^2} = 0$

$x = 0, e^{-x^2} \neq 0$

$h'(-1) = -2(-1)e^{-(-1)^2} \approx .736$

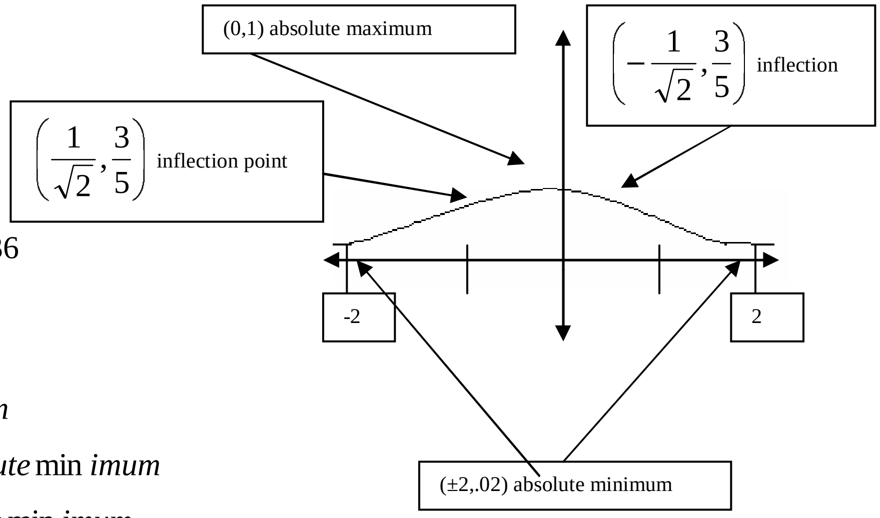
$h'(1) = -2(1)e^{-(1)^2} \approx -.736$

$h(0) = e^{-0^2}$

$h(0) = 1, \textit{absolute} \max \textit{imum}$

$h(-2) = e^{-(-2)^2} = .02, \textit{absolute} \min \textit{imum}$

$h(2) = e^{-(2)^2} = .02, \textit{absolute} \min \textit{imum}$



The graph above was created with the drawing features of *Word*, so it represents a poor sketch. Using graph paper, the student should be able to reproduce a better sketch.

$h'(x) = -2xe^{-x^2}$

$h''(x) = -2 \cdot e^{-x^2} + -2x \cdot -2xe^{-x^2}$

$h''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$

$h''(x) = 2e^{-x^2}\left(-1 + 2x^2\right)$

$2e^{-x^2}\left(-1 + 2x^2\right) = 0$

$-1 + 2x^2 = 0, 2e^{-x^2} \neq 0$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

$h''(-2) = 2e^{-(-2)^2}\left(-1 + 2(-2)^2\right) = -.256$

$h''(0) = 2e^{-(0)^2}\left(-1 + 2(0)^2\right) = 2$

$h''(2) = 2e^{-(2)^2}\left(-1 + 2(2)^2\right) = -.256$

$h\left(\frac{1}{\sqrt{2}}\right) = e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = .6$

$h\left(-\frac{1}{\sqrt{2}}\right) = e^{-\left(-\frac{1}{\sqrt{2}}\right)^2} = .6$

#6 Describe the graph of $f(x) = x^4 - 2x^2$.

$f(x) = x^4 - 2x^2$
$f'(x) = 4x^3 - 4x$
$4x^3 - 4x = 0$
$4x(x^2 - 1) = 0$
$x = 0,\ x = 1,\ x = -1$

> Extrema can occur where the derivative is undefined and/or equal to zero. Begin by finding derivative. Set derivative equal to zero and solve. Zero, positive one, and negative one are critical numbers and possibly extrema.

$f'(-2) = -24$
$f'(-½) = 1.5$
$f'(½) = -1.5$
$f'(2) = 24$

> Since the derivative is negative left of negative one and between zero and one, the function decreases from (–∞,–1)U(0,1). Since the derivative is positive between negative one and zero and right of one, the function increases (–1,0)U(1,∞).

$f(-1) = -1$
$f(1) = -1$
$f(0) = 0$

> Since the function decreases from (–∞,–1) then increases (–1,0), a relative minimum occurs when $x = -1$. Therefore, $f(-1) = -1$ is a relative minimum. Likewise, another minimum occurs at $x = 1$, so $f(1) = -1$ is also a relative minimum. Incidentally, these two extrema are the absolute minimum values of the function. Since the function increases from (–1,0) then decreases from (0,1), a relative maximum occurs when $x = 0$. Therefore, $f(0) = 0$ is a relative maximum.

$f''(x) = 12x^2 - 4$
$12x^2 - 4 = 0$
$3x^2 - 1 = 0$
$3x^2 = 1$
$x^2 = ⅓$
$x = \pm\frac{1}{\sqrt{3}}$

> The critical numbers of the first derivative (where the second derivative is undefined or equal to zero) represent possible inflection points.

$f''(-1) = 8$
$f''(0) = -4$
$f''(1) = 8$
$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{5}{9}$
$f\left(\frac{1}{\sqrt{3}}\right) = -\frac{5}{9}$

> Since the second derivative is positive for x-values left of $-\frac{1}{\sqrt{3}}$ and right of $\frac{1}{\sqrt{3}}$ but negative in between these two x-values, the function is concave up everywhere except along $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ where it is concave down. Since there is a change in concavity, inflection occurs when $x = \pm\frac{1}{\sqrt{3}}$.

ANSWER for #6: The function increases on the interval (–1,0)U(1,∞) and decreases on the interval (–∞,–1)U(0,1). The function has absolute minimums at the points (–1, –1) and (1, –1). The function has a relative maximum at the point (0,0). The function is concave down on the interval $\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$**. The function is concave up on the interval** $\left(-\infty,-\frac{1}{\sqrt{3}}\right)\cup\left(\frac{1}{\sqrt{3}},\infty\right)$**. Inflection occurs at the points** $\left(-\frac{1}{\sqrt{3}},-\frac{5}{9}\right)$ **and** $\left(\frac{1}{\sqrt{3}},1\frac{5}{9}\right)$**. The intercepts occur at** $\left(-\sqrt{2},0\right), (0,0), \left(\sqrt{2},0\right)$.

#7 Sketch the graph of $h(x) = x^2 - \ln x$

Find the restrictions:
$x > 0$, because the argument of a logarithm must be positive.

Find asymptotes:
Logarithmic functions have vertical asymptotes that act as a boundary to the domain. The vertical asymptote is $x = 0$.

Find the y-intercept:
The graph does not intersect the y-axis because $x > 0$.

Find the x-intercepts:

$x^2 - \ln x = 0$

$\ln x = x^2$

$e^{x^2} \neq x$

The graph does not intersect the x-axis.

Find behavior:

$$h(x) = x^2 - \ln x$$

$$h'(x) = 2x - \frac{1}{x}$$

$$2x - \frac{1}{x} = 0$$

$$x \cdot \left(2x - \frac{1}{x}\right) = 0 \cdot x$$

$$2x^2 - 1 = 0$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{1}{\sqrt{2}}$$

$$h'(.5) = -12 < 0$$

$$h'(1) = 1 > 0$$

$$h(x)\, decreases \left(0, \frac{1}{\sqrt{2}}\right)$$

$$h(x)\, increases \left(\frac{1}{\sqrt{2}}, \infty\right)$$

Find extrema:

This function possesses an *absolute minimum* at its critical value.

$$h\left(\frac{1}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^2 - \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$h\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} - \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{2} - \ln\left(\frac{1}{\sqrt{2}}\right)\right)$$

Find concavity:

$$h'(x) = 2x - \frac{1}{x}$$

$$h''(x) = 2 + \frac{1}{x^2}$$

$$2 + \frac{1}{x^2} > 0$$

concave up

$(0, \infty)$

Find inflection:

This function does not possess any inflection.