

**POLYNOMIAL FUNCTIONS**

Examples of polynomial functions:

These examples are discussed below.

$$\left\{ \begin{array}{l} f(x) = x^5 + 4x^4 - 11x^3 - 16x^2 + 58x - 60 \\ g(x) = 5 - x^2 \\ h(x) = 4x^2 + 12x + 9 \\ d(t) = -t^2 - 9 \\ j(k) = 2(k + 1)^2(k - 3)^3 \\ v(x) = x^3 - 8 \\ w(x) = x^2 + 6x + 8 \end{array} \right.$$

**INTRODUCTORY TERMS**

**Real Numbers** – Real numbers include all rational and irrational numbers.

**Rational Numbers** – rational numbers are numbers that can be written in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are integers and  $b \neq 0$  (i.e., the fractions). In their decimal form, rational numbers repeat or terminate. For example,  $\frac{1}{3}$  can be written  $\overline{.3}$  with the three repeating, and  $\frac{1}{2}$  can be written  $.5$  terminating in the tenths place. Rational numbers include numbers like 1,  $-2$ , 15,  $\frac{1}{2}$ ,  $\frac{2}{3}$ , and  $-5\frac{1}{3}$ .

**Irrational Numbers** – irrational numbers are numbers that are not rational, i.e., with the above definition, irrational numbers do *not* terminate or repeat in their decimal form. Irrational numbers include numbers like  $\sqrt{2}$ ,  $1 + 2\sqrt{3}$ ,  $7 - \sqrt{5}$ , and  $\pi$ .

**Complex Numbers** – complex numbers are numbers of the form  $a + bi$  where  $a$  and  $b$  are real numbers, and  $i = \sqrt{-1}$ . In other words, complex numbers include numbers with imaginary parts. Imaginary numbers originate from taking the square root (or any even root) of a negative number. Examples of complex numbers include  $-i$ ,  $2 + 3i$ ,  $1 - 2i$ .

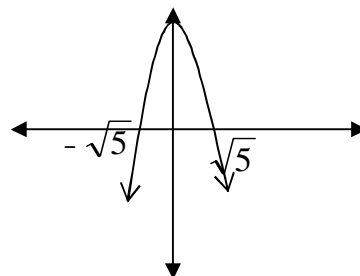
**CRUCIAL TERMS**

**Roots (also called Zeros & Solutions)** – the  $x$ -values of a polynomial function that give it a  $y$ -value equal to zero represent the roots of the polynomial. In symbols, for  $P(x)$ , if  $P(x_1) = 0$ , then  $x_1$  is a root. For instance, if  $P(2) = 0$ , then 2 is a root. Using the polynomial functions above,  $-2$  is a root of  $w(x)$  because  $w(-2) = 0$ :

$$\begin{aligned} w(x) &= x^2 + 6x + 8 \\ w(-2) &= (-2)^2 + 6(-2) + 8 \\ w(-2) &= 4 - 12 + 8 \\ w(-2) &= -8 + 8 \\ w(-2) &= 0. \end{aligned}$$

Roots can be real or complex. If they are real, roots can be positive or negative, and they can be rational or irrational. Real roots correspond with  $x$ -intercepts on the graph of the polynomial. Solving the polynomial derives its roots. For example, solve  $g(x)$  by setting the function equal to zero:

$$\begin{aligned} 5 - x^2 &= 0 \\ -x^2 &= -5 \\ x^2 &= 5 \\ \sqrt{x^2} &= \sqrt{5} \\ x &= \pm\sqrt{5} \end{aligned}$$



The roots of  $g(x)$  are  $\sqrt{5}$  and  $-\sqrt{5}$ . Accordingly,  $g(x)$  has  $x$ -intercepts at  $(\sqrt{5}, 0)$  and  $(-\sqrt{5}, 0)$ .  $\sqrt{5}$  and  $-\sqrt{5}$

are irrational roots approximately equal to 2.236 and  $-2.236$ . Solving  $d(t)$  similarly, yields two complex roots:

$$-t^2 - 9 = 0$$

$$-t^2 = 9$$

$$t^2 = -9$$

$$\sqrt{t^2} = \sqrt{-9}$$

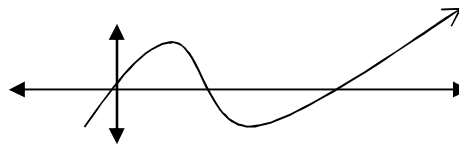
$$t = \pm 3i$$

The roots then of  $d(t)$  are  $3i$  and  $-3i$ . These are complex since they have imaginary parts. Complex roots do *not* appear as  $x$ -intercepts on the graph of the polynomial. *All roots correspond with **linear factors** (see below).*

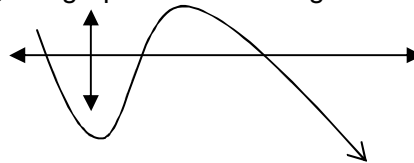
**Constant** – the constant is the term of a polynomial without a variable part. In the examples above, the constant of  $f(x)$  has a constant equal to  $-60$ , and  $g(x)$  has a constant equal to  $5$ . The constant for  $j(k)$  is not as obvious because  $j(k)$  is not written in expanded form. We can determine the constant without expanding the polynomial thusly:  $2 \cdot 1^2 \cdot 3^3 = 54$ , so  $54$  is the constant of  $j(k)$ . The constant corresponds to the  $y$ -intercept on the graph of the polynomial. Thus,  $f(x)$  with a constant of  $-60$  crosses the  $y$ -axis at  $(0, -60)$ .

**Degree (or Order)** – the highest exponent determines the degree, denoted by  $n$ , of a polynomial. The degree of  $f(x)$  is  $5$  ( $n = 5$ ). For both  $g(x)$  and  $h(x)$ , the degree is  $2$  ( $n = 2$ ). Second-degree polynomials are called quadratics. The degree of  $v(x)$  is  $3$  ( $n = 3$ ). Third-degree polynomials are called cubics. The degree corresponds to the number of roots possessed by a polynomial. Thus,  $f(x)$  with a degree of  $5$  has five roots.

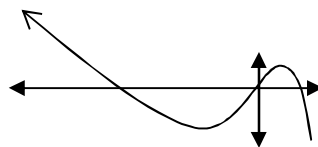
**End Behavior** – end behavior describes the behavior of the graph to the right (as the  $x$ -values approach infinity) and to the left (as the  $x$ -values approach negative infinity). End behavior is described as either rising or falling. For right-end behavior, if the function increases as the  $x$ -values increase, the graph rises on the right:



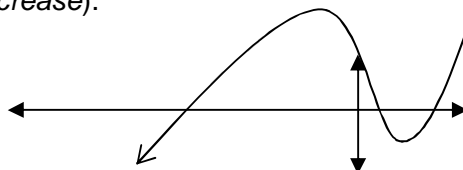
If the function decreases as  $x$ -values increase, the graph falls on the right:



For left-end behavior, if the function decreases as  $x$ -values increase, the graph rises toward the left (in other words, the function *increases* as  $x$ -values *decrease*):



If the function increases as  $x$ -values increase, the graph falls toward the left (in other words, the function *decreases* as  $x$ -values *decrease*):



**Imaginary Roots (or Complex Roots)** – imaginary roots are complex numbers (see above) that give the function a value of zero. Imaginary roots always occur in conjugate pairs:  $a + bi$  and  $a - bi$ . Imaginary roots

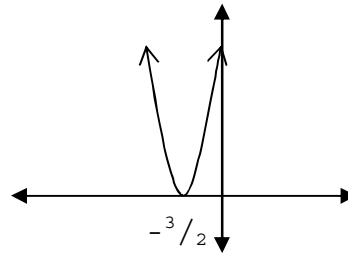
occur in pairs because they are derived from the square root of a negative number (as a result of the quadratic formula), which is preceded by the positive/negative sign ( $\pm$ ).

**Leading Coefficient** – the leading coefficient is the coefficient to the variable with the highest exponent. The leading coefficient of  $f(x)$  is 1. For  $g(x)$  the leading coefficient is  $-1$ . For  $h(x)$  the leading coefficient is 4. For  $j(k)$  the leading coefficient is 2.

**Linear Factors** – the linear factors are the prime factors (allowing for factors with complex numbers) of a polynomial function. Linear factors correspond with the roots of a polynomial (see above). The factors of  $w(x)$  are  $(x + 2)(x + 4)$  because the product of these two binomials equals  $x^2 + 6x + 8$ . For another example, consider  $j(k)$ . Since  $j(k)$  is written in factored form, its linear factors,  $(k + 1)$  and  $(k - 3)$ , are easily identified. Each factor corresponds to a root of  $j(k)$ . The root of  $(k + 1)$  is  $-1$ . The root of  $(k - 3)$  is 3. Since each of these factors is repeated,  $(k + 1)$  twice and  $(k - 3)$  thrice, the roots possess **multiplicity** (see below). The root  $-1$  has multiplicity two because it comes from a factor repeated twice. The root 3 has multiplicity three because it comes from a factor repeated thrice.

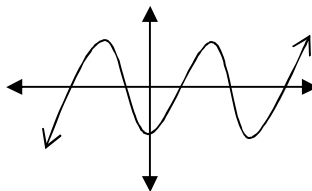
**Multiplicity** – multiplicity, denoted by  $m$ , is the number of repetitions of a given root. For example,  $h(x)$  has a repeated root. Its root comes from the repeated linear factor  $(2x + 3)$ . Since this linear factor is repeated twice, its corresponding root  $-\frac{3}{2}$  has multiplicity of two:

$$\begin{aligned} 4x^2 + 12x + 9 &= 0 \\ (2x + 3)(2x + 3) &= 0 \\ 2x + 3 &= 0 \\ 2x &= -3 \\ x &= -\frac{3}{2} \quad m2 \end{aligned}$$



Multiplicity determines a polynomial function's behavior near the  $x$ -axis. If the root has even multiplicity, the function will bounce off the  $x$ -axis as shown in the graph above. If a root has odd multiplicity, the function will cut through the  $x$ -axis.

**Turning Points** – a turning point is a change in functional behavior from increasing to decreasing or vice versa. If a polynomial has  $n$ -distinct roots, it will have  $n - 1$  turning points. In other words, if a polynomial has 5 distinct roots, it will change increasing/decreasing behavior four times:



Turning points comprise a subject for study using the calculus. In College Algebra, the student must simply be cognizant of their existence.

### FUNDAMENTAL THEOREMS

**Factor Theorem:** *If  $r$  is a root of the polynomial  $P(x)$ , then  $(x - r)$  is a factor of  $P(x)$ . Conversely, if  $(x - r)$  is a factor of  $P(x)$ , then  $r$  is a root of  $P(x)$ .* This theorem simply refers to the relation between linear factors and roots.

**Fundamental Theorem of Algebra:** *Every polynomial  $P(x)$  of degree  $n > 0$  has at least one root.* This theorem asserts that all polynomial functions have at least one root.

**$N$  Roots Theorem:** *Every polynomial  $P(x)$  of degree  $n > 0$  can be expressed as the product of  $n$  linear factors. Hence,  $P(x)$  has exactly  $n$  roots—not necessarily distinct.* This theorem says that the degree of the polynomial determines how many linear factors (including factors containing complex numbers) and how many roots the polynomial possesses. The roots may not be distinct, meaning the roots can be repeated (multiplicity).

**Remainder Theorem:** *If  $R$  is the remainder after dividing the polynomial  $P(x)$  by  $x - r$ , then  $P(r) = R$ .* This theorem tells us that the remainder of the division algorithm is the  $y$ -value coordinated with the divisor. For example, divide  $w(x)$  synthetically by 1:

$$\begin{array}{r|rrrr} 1 & 1 & 6 & 8 & \\ & & & 1 & 7 \\ \hline & & & & 15 \end{array}$$

The remainder, 15, is the  $y$ -value coordinated with the divisor, 1. Consequently, according to the Remainder Theorem,  $w(1) = 15$ . This theorem is important because it reveals a method for finding roots discussed below.

**Imaginary Roots Theorem:** *Imaginary roots of polynomials with real coefficients, if they exist, occur in conjugate pairs.* Since this course deals only with polynomials with real coefficients, this theorem tells us that complex roots occur in conjugate pairs. If a polynomial has a complex root  $a + bi$ , it will have a complex root  $a - bi$ .

**Real Roots and Odd-Degree Polynomial Theorem:** *A polynomial of odd degree with real coefficients always has at least one real root.* This theorem tells us that odd-degree polynomials have at least one real root.

**Rational Root Theorem:** *If the rational number  $\frac{b}{c}$ , in lowest terms, is a root of the polynomial,  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  with integer coefficients, then  $b$  must be an integer factor of  $a_0$  and  $c$  must be an integer factor of  $a_n$ .* This theorem tells us that any rational roots of a polynomial must be some quotient of the factors of the constant and leading coefficient. The Rational Root Theorem is an important tool in solving polynomials because it quickly yields a manageable set of *possible* rational roots. The Remainder Theorem then determines whether a possible rational root is in fact a root.

**Root Location Theorem:** *If  $P(x)$  is a polynomial function defined in such a way that for real numbers  $a$  and  $b$ ,  $P(a)$  and  $P(b)$  differ in sign, then there exists at least one real root between  $a$  and  $b$ .* This theorem tells us that a polynomial function must have a real root between two  $x$ -values whose coordinates have opposite signs.

**Descartes' Rule of Sign:** *If  $P(x)$  defines a polynomial function with real coefficients and a nonzero constant, with terms in descending powers of  $x$ , the number of positive real roots of  $P(x)$  either equals the number of variations in sign occurring in the coefficients of  $P(x)$  or is less than the number of variations by a multiple of two. Moreover, the number of negative real roots of  $P(x)$  either equals the number of variations in sign occurring in the coefficients of  $P(-x)$  or is less than the number of variations by a multiple of two.* This theorem tells us that the number of positive roots of a polynomial equals the number of sign changes in the coefficients if the polynomial is written in proper descending order and has a nonzero constant.

### LECTURE

The attached lecture models a 10-step process for graphing polynomial functions. The process uses an elementary strategy for finding the roots. The lecture uses  $f(x) = x^5 + 4x^4 - 11x^3 - 16x^2 + 58x - 60$  as an example. While  $f(x)$  is a typical polynomial, it lacks some characteristics. For example,  $f(x)$  does not have any repeated factors; consequently, it lacks roots with multiplicity. While using synthetic division to find roots, the student should keep in mind that any rational root can be repeated.

**POLYNOMIAL FUNCTIONS**

$$f(x) = x^5 + 4x^4 - 11x^3 - 16x^2 + 58x - 60$$

- The constant tells us the y-intercept (0,-60).
- The sign of the leading coefficient tells us the right-end behavior.
  - \* Positive leading coefficient = rising on the right [ $f(x)$  is increasing on right ↗].
  - \* Negative leading coefficient = falling on right.
- The degree's "evenness" or "oddness" tells us the left-end behavior in comparison to the right-end behavior. \*
 

Even degree = same behavior. Odd degree = opposite behavior (falling on left ↘).
- The degree also tells us the number of roots [also called zeros or solutions] (5).
- The sign changes tell us the number of positive roots (3). [ $f(x)$  has 3 or 1 positive real roots]
 

\* Note: The number of real roots can be reduced by multiples of two since some roots may be imaginary and imaginary roots always come in pairs.
- The sign changes of  $f(-x)$  tell us the number of negative roots (2). [ $f(x)$  has 2 or none negative real roots]
 
$$f(-x) = (-x)^5 + 4(-x)^4 - 11(-x)^3 - 16(-x)^2 + 58(-x) - 60$$

$$f(-x) = -x^5 + 4x^4 + 11x^3 - 16x^2 - 58x - 60$$
- The ratios derived by placing the factors of the constant over the factors of the leading coefficient give us the possible rational roots:
 
$$\pm \frac{60}{1} = \pm \frac{1,2,3,4,5,6,10,12,15,20,30,60}{1} = \pm 1,2,3,4,5,6,10,12,15,20,30,60$$

\*Note: If there were more factors of the leading coefficient, then the possible rational roots would include fractions.
- Synthetic division with the possible rational roots will eventually yield a rational root (if there is one).

$$\begin{array}{r|rrrrrr} 6 & 1 & 4 & -11 & -16 & 58 & -60 \\ & & 6 & 60 & 294 & 1668 & 10,356 \\ \hline & 1 & 10 & 49 & 278 & 1726 & 10,296 \end{array}$$

(NO ROOTS LARGER THAN SIX) →

$$\begin{array}{r|rrrrrr} -6 & 1 & 4 & -11 & -16 & 58 & -60 \\ & & -6 & 12 & -6 & 132 & -1140 \\ \hline & 1 & -2 & 1 & -22 & 190 & -1146 \end{array}$$

(NO ROOTS LEFT OF NEGATIVE SIX) →

$$\begin{array}{r|rrrrrr} 1 & 1 & 4 & -11 & -16 & 58 & -60 \\ & & 1 & 5 & -6 & -22 & -36 \\ \hline & 1 & 5 & -6 & -22 & 36 & -24 \end{array}$$

$$\begin{array}{r|rrrrrr} 3 & 1 & 4 & -11 & -16 & 58 & -60 \\ & & 3 & 21 & 30 & 42 & 300 \\ \hline & 1 & 7 & 10 & 14 & 100 & 240 \end{array}$$

$$\begin{array}{r|rrrrrr} 2 & 1 & 4 & -11 & -16 & 58 & -60 \\ & & 2 & 12 & 2 & -28 & 60 \\ \hline & 1 & 6 & 1 & -14 & 30 & 0 \end{array}$$

$$\begin{array}{r|rrrrrr} -5 & 1 & 6 & 1 & -14 & 30 & 0 \\ & & -5 & -5 & 20 & -30 & \\ \hline & 1 & 1 & -4 & 6 & 0 & \end{array} \quad x = 2$$

$$\begin{array}{r|rrrrrr} -3 & 1 & 1 & -4 & 6 & 0 & \\ & & -3 & 6 & -6 & & \\ \hline & 1 & -2 & 2 & 0 & & \end{array} \quad x = -5$$

$$\begin{array}{r|rrrrrr} & 1 & -2 & 2 & 0 & & \\ & & & & & & \\ \hline & 1 & -2 & 2 & 0 & & \end{array} \quad x = -3$$

$$x^2 - 2x + 2 = \text{quadratic factor}$$

- A The Upper Bound Theorem tells us that if a positive divisor yields an all-positive or all-negative quotient, then the divisor represents an upper bound, meaning there are no roots to the right of the divisor.
- B The Lower Bound Theorem tells us that if a negative divisor yields a quotient with alternating signs, then the divisor is a lower bound, meaning there are no roots to the left of the divisor.
- C The Remainder Theorem tells us that the function passes through (1,-24). In other words  $f(1)=-24$ .
- D The Root Location Theorem tells us there is a root between 1 and 3 because there was a change in the sign of the remainder with these two divisors.
- E Remainder Theorem tells us 2 is a root (2,0).

**POLYNOMIAL FUNCTIONS**

9. The quadratic formula will yield the final two roots.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x^2 - 2x + 2$$

$$a = 1 \quad b = -2 \quad c = 2$$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$x = \frac{2 \pm \sqrt{-4}}{2}$$

$$x = \frac{2 \pm 2i}{2}$$

$$x = \frac{2(1 \pm i)}{2}$$

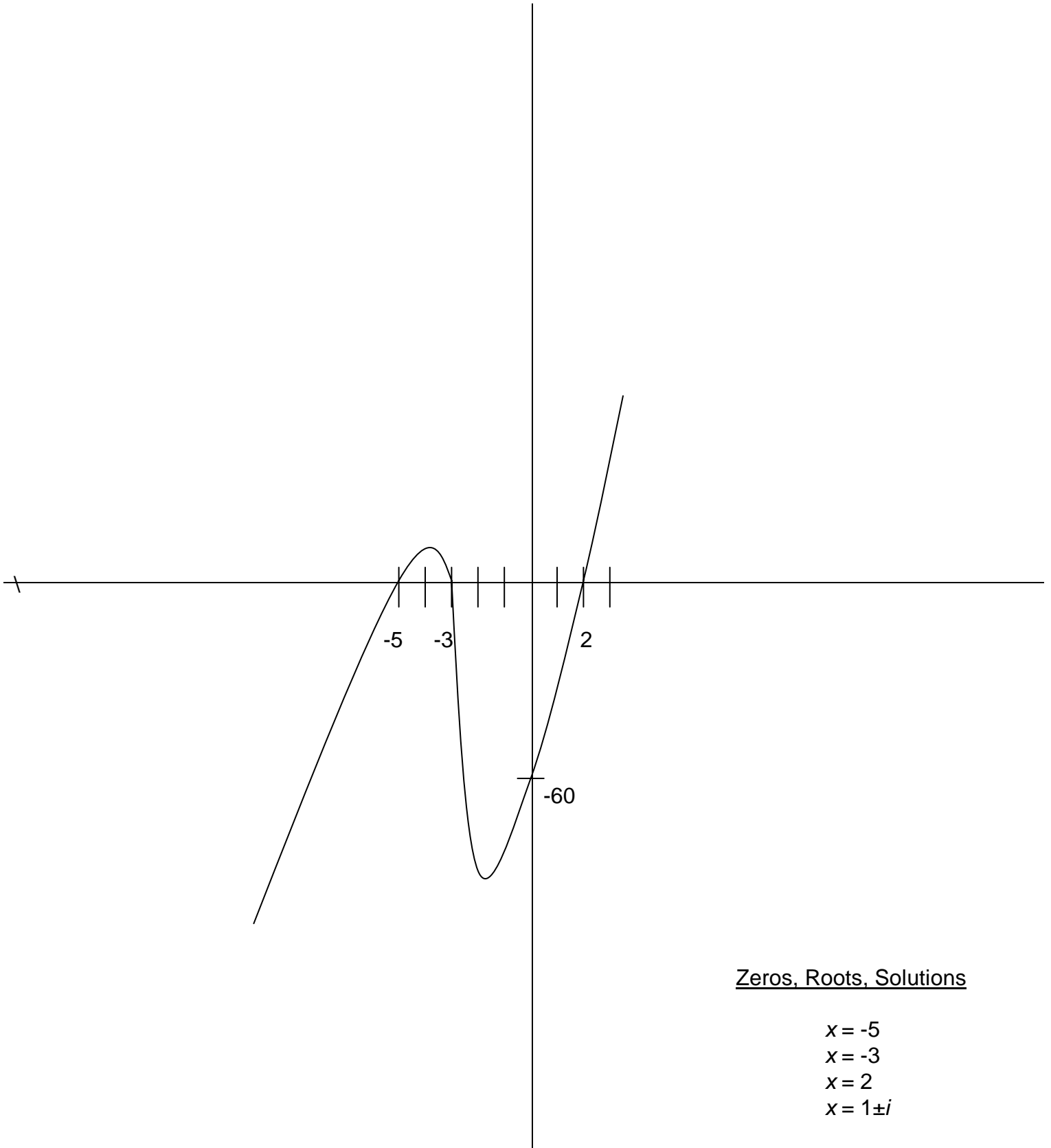
$$x = 1 \pm i$$

The roots/zeros are:  $1+i$ ,  $1-i$ ,  $2$ ,  $-5$ ,  $-3$

The linear factors are:  $(x-1-i)(x-1+i)(x-2)(x+5)(x+3)$

10. The multiplicity of each root tells us if the function cuts or bounces off the x-axis at the given root. Odd multiplicity = cut. Even multiplicity = bounce. All of the roots appeared only once giving them each odd multiplicities meaning they all cut the x-axis. It is important to note, however, that sometimes a root may have greater multiplicity.

# POLYNOMIAL FUNCTIONS



## Zeros, Roots, Solutions

$$x = -5$$

$$x = -3$$

$$x = 2$$

$$x = 1 \pm i$$

